Electric Field of charged hollow and solid Spheres

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Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss’ Law for a cylindrical charge
- Gauss’ Law for a charged plane
- Laplace’s and Poisson’s Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

Downloads: www.demul.net/frits, scroll to “Electricity and Magnetism”
Available:
A sphere, radius $R$, with
- surface charge density $\sigma$ [C/m$^2$], or
- volume charge density $\rho$ [C/m$^3$]

$\sigma$ and $\rho$ can be $f$ (position)

Question:
Calculate $E$-field in arbitrary points inside and outside the sphere
Contents:
1. A hollow sphere, homogeneously charged (conducting)
2. Idem., non-homogeneously charged
3. A solid sphere, homogeneously charged
4. Idem., non-homogeneously charged

Method: integration over charge elements (radial and angular integrations).

NB. Homogeneously charged spheres can also be calculated in an easy way using the symmetry in Gauss’ Law.
Contents:

1. A hollow sphere, homogeneously charged (conducting)
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1. Hollow sphere, homogeneously charged

1. Charge distribution: (surface charge) $\sigma \text{[C/m}^2\text{]}$

2. Coordinate axes:
   
   $Z$-axis = polar axis

3. Symmetry: spherical

4. Spherical coordinates:
   
   $r, \theta, \varphi$
1. Hollow sphere, homogeneously charged

1. XYZ-axes

2. Point P on Z-axis

3. all $Q_i$’s at $r_i$, $\theta_i$, $\varphi_i$
   contribute $E_i$ to $E$ in P

4. In P: $E_{i,x}$, $E_{i,y}$, $E_{i,z}$

5. Expect from symmetry:
   $$\sum (E_{i,x}+E_{i,y}) = 0$$

6. $E = E_z e_z$ only!
1. Hollow sphere, homogeneously charged

Distributed charge: \( dQ \)

\[
dE_z = \frac{dQ}{4\pi\varepsilon_0 r^2} (e_r \cdot e_z)
\]

charge element

\( dQ = \sigma dA \)

surface element

\( dA = (R.d\theta).(R.\sin\theta.d\varphi) \)

\( r \) and \((e_r \cdot e_z)\):

see next page
1. Hollow sphere, homogeneously charged

Cross section through OP:

\[ r^2 = (R \cdot \sin \theta)^2 + (z_p - R \cdot \cos \theta)^2 \]

\[ (e_r \cdot e_z) = \frac{(z_p - R \cdot \cos \theta)}{r} \]
1. Hollow sphere, homogeneously charged

\[ dE_z = \frac{\sigma \ dA}{4\pi \varepsilon_0 r^2} (e_r \cdot e_z) \]

\[ dA = (R \cdot d\theta) \cdot (R \cdot \sin \theta \cdot d\phi) \]

\[ r^2 = (R \cdot \sin \theta)^2 + (z_p - R \cdot \cos \theta)^2 \]

\[ (e_r \cdot e_z) = \frac{(z_p - R \cdot \cos \theta)}{r} \]

These expressions can be used for both \(|z_p| \geq R\) and \(|z_p| < R\)

\[ dE_z = \frac{\sigma \ R \cdot d\theta \cdot R \cdot \sin \theta \cdot d\phi \cdot [z_p - R \cdot \cos \theta]}{4\pi \varepsilon_0 \left[ (R \cdot \sin \theta)^2 + (z_p - R \cdot \cos \theta)^2 \right]^{3/2}} \]

\[ E_z = \int_0^\frac{\pi}{2} d\theta \int_0^{2\pi} d\phi \frac{\sigma \ R^2 \cdot \sin \theta \cdot [z_p - R \cdot \cos \theta]}{4\pi \varepsilon_0 \left[ (R \cdot \sin \theta)^2 + (z_p - R \cdot \cos \theta)^2 \right]^{3/2}} \]
1. Hollow sphere, homogeneously charged

The electric field of a charged hollow or solid sphere can be described by the following integral expression:

\[ E_z = \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} d\varphi \, d\theta \, \sigma \, \frac{R^2 \sin \theta \left[ z_p - R \cos \theta \right]}{4\pi \varepsilon_0 \left[ (R \sin \theta)^2 + (z_p - R \cos \theta)^2 \right]^{3/2}} \]

Integration:
- over \( \varphi \) : results in factor \( 2\pi \)
- over \( \theta \) : replace \( \sin \theta \, d\theta \) by \( -d(\cos \theta) \), divide above and below by \( R^3 \), call \( \cos \theta = x \), and \( z_p/R = a \)
- and \( a^2 + 1 - 2ax = y \) (with \( -2a \, dx = dy \))
- and integrate over \( y \)

Result of integration:

\[ E_z = \frac{\sigma F}{2\varepsilon_0} \text{, with } F = \frac{1}{2} \left[ \frac{-1}{\sqrt{y}} + \frac{1}{a^2 \sqrt{y}} + \frac{\sqrt{y}}{a^2} \right]_{\theta=0}^{\theta=\pi} \]

2 cases: \(|a| > 1 \) and \(|a| < 1 \).

This expression can also be used for other limiting values of \( \theta \).
1. Hollow sphere, homogeneously charged

\[ E_z = \frac{\pi}{2} \int_{0}^{2\pi} d\phi \int_{0}^{\theta_b} d\theta \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2 \sin \theta \left [ z_p - R \cos \theta \right ]}{\left [ (R \sin \theta)^2 + (z_p - R \cos \theta)^2 \right ]^{3/2}} \]

Result of integration:

\[ E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \frac{1}{2} \left [ \frac{-1}{\sqrt{y}} + \frac{1}{a^2 \sqrt{y}} + \frac{\sqrt{y}}{a^2} \right ]_{\theta=\pi}^{\theta=0} \]

with \( a = z_p / R \) and \( y = a^2 + 1 - 2a \cos \theta \).

2 cases: \( |a| > 1 \) and \( |a| < 1 \).

This expression can also be used for other limiting values of \( \theta \).

Next slide: plot of \( F \) for varying values of the integration limits:
begin \( \theta_b = 0..180^0 \); end \( \theta_e = 180^0 \)

\( \theta_b = 0^0 \): closed sphere
\( \theta_b > 0^0 \): “bowl” shape
1. Hollow sphere, homogeneously charged

\[ E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \frac{1}{2} \left[ \frac{-1}{\sqrt{y}} + \frac{1}{a^2 \sqrt{y}} + \sqrt{y} \right]^{\theta=\pi}_{\theta=0} \]

with \( a = z_P/R \) and \( y = a^2 + 1 - 2a \cos \theta \).

Plot of \( F \) for varying values of the integration limits:
begin \( \theta_b = 0..180^0 \); end \( \theta_e = 180^0 \)

\( \theta_b = 0^0 \) : closed sphere
\( \theta_b > 0^0 \) : “bowl” shape
In plot: \( \theta_b = “thb” \)

1st curve, “2/a^2” :
\[ E = \frac{\sigma}{2\varepsilon_0} \frac{2}{a^2} = \frac{\sigma}{\varepsilon} \frac{R^2}{z_P^2} \]

= according to “Gauss”

E-field of a charged hollow or solid sphere
1. Hollow sphere, homogeneously charged

\[ E_z = \frac{\sigma F}{2\varepsilon_0} , \text{ with } F = \frac{1}{2} \left[ \frac{-1}{\sqrt{y}} + \frac{1}{a^2 \sqrt{y}} + \sqrt{y} \right]^{\theta=\pi}_{\theta=0} \]

Correspondence: for \( \theta_b=0 \) with result from “Gauss”.
Result = 0 inside sphere

1st curve, “2/a^2” (= “Gauss”) (abs.value)

\( \theta_b=90^0: \) Symmetric around \( z=0 \) for “half sphere”

\( \theta_b=30^0 \text{ and } 150^0: \)
\( \theta_b=60^0 \text{ and } 120^0: \)
mutual symm. around \( z=0 \) inside sphere

\[ a = -1 .. +1 : \]
inside sphere:
\[ z = -R .. +R \]

E-field of a charged hollow or solid sphere
1. Hollow sphere, homogeneously charged

\[ E_z = \frac{\sigma F}{2\varepsilon_0} \]

with \[ F = \frac{1}{2} \left[ \frac{-1}{\sqrt{y}} + \frac{1}{a^2 \sqrt{y}} + \frac{\sqrt{y}}{a^2} \right] \]

\[ \theta = \pi \]

with \( a = \frac{z_0}{R} \) and \( y = a^2 + 1 - 2a \cdot \cos \theta \).

Plot of \( F \) for integration limits:
begin \( \theta_b = 45 \); end \( \theta_e = 135^0 \) (ring)

\[ a = -1 \ldots +1 \] : inside sphere:
\[ z = -R \ldots +R \]
1. Hollow sphere, homogeneously charged

\[ E_z = \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2 \sin \theta [z_p - R \cos \theta]}{[(R \sin \theta)^2 + (z_p - R \cos \theta)^2]^{3/2}} \]

Result for \( E \) in \( P \):

- \( z_p < R \): \( E = 0 \)
- \( z_p > R \): \( E = \frac{\sigma R^2}{\varepsilon_0 z_p} \) \( e_z = \frac{Q}{4\pi\varepsilon_0 z_p^2} e_z \)

Using \( \sigma = \frac{Q}{4\pi R^2} \).

With Gauss:

Gauss sphere at radius \( z_p \):

\[ \iint E \, dA = \frac{Q_{encl}}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 z_p^2} \]

E-field of a charged hollow or solid sphere
Contents:
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4. Idem., non-homogeneously charged
2. Hollow sphere, non-homogeneously charged

Now \( \sigma = f(\theta, \varphi) \)

\[
E_z = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi \varepsilon_0} \frac{R^2 \sin \theta [z_p - R \cos \theta]}{[(R \sin \theta)^2 + (z_p - R \cos \theta)^2]^{3/2}}
\]

\( X \) and \( Y \)-components may be present; \( e_x \) and \( e_y \) are \( \perp e_z \)

For \( E_z \): \( (e_r \cdot e_z) = (z_p - R \cos \theta) / r \)

For \( E_x \) and \( E_y \) we need: \( R \sin \theta / r \)
2. Hollow sphere, non-homogeneously charged

Now $\sigma = f(\theta, \varphi)$

$$E_z = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2 \sin \theta [z_p - R \cos \theta]}{[(R \sin \theta)^2 + (z_p - R \cos \theta)^2]^{3/2}}$$

For $E_x$ and $E_y$ we need: $R \sin \theta / r$

$$E_{x,y} = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2 \sin \theta [R \sin \theta \Phi(\varphi)]}{[(R \sin \theta)^2 + (z_p - R \cos \theta)^2]^{3/2}}$$

with $\Phi(\varphi) = \cos \varphi$ (for $E_x$); $\Phi(\varphi) = \sin \varphi$ (for $E_y$).

In many situations, e.g. asymmetric charge distributions, numerical integration will be necessary.
Contents:

1. A hollow sphere, homogeneously charged (conducting)
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3. Solid sphere, homogeneously charged

Derived for a **hollow** sphere:

\[
E_z = \frac{\pi}{4\pi \varepsilon_0} \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{\sigma}{[R \sin \theta]^2 + (z_p - R \cos \theta)^2]^{3/2}} \frac{R^2 \sin \theta (z_p - R \cos \theta)}{\sqrt{(R \sin \theta)^2 + (z_p - R \cos \theta)^2}}
\]

For a **solid** sphere we need an **extra integration variable**: over varying radius.

Suppose: radius \( R \) varies from 0 to \( R_0 \).

Integration element \( dR \)

**Surface charge element**

\[
dQ = \sigma \, dA = \sigma \, R \, d\theta \, R \sin \theta \, d\phi \quad (\sigma \text{ in } \text{C/m}^2)
\]

has to be replaced by

**Volume charge element**: \( (\rho \text{ in } \text{C/m}^3) \)

\[
dQ = \rho \, dV = \rho \, dR \, R \, d\theta \, R \sin \theta \, d\phi
\]

\[
E_z = \int_0^{R_0} dR \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\rho}{4\pi \varepsilon_0} \frac{R^2 \sin \theta [z_p - R \cos \theta]}{[R \sin \theta]^2 + (z_p - R \cos \theta)^2]^{3/2}}
\]
3. Solid sphere, homogeneously charged

$$E_z = \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^{R_0} dR \frac{\rho}{4\pi \varepsilon_0} \frac{R^2 \sin \theta \left[ z_p - R \cos \theta \right]}{\left[ (R \sin \theta)^2 + (z_p - R \cos \theta)^2 \right]^{3/2}}$$

Integration over $\theta$ and $\phi$ resulted in:

$$E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \frac{1}{2} \left[ -\frac{1}{\sqrt{y}} + \frac{1}{a^2 \sqrt{y}} + \frac{\sqrt{y}}{a^2} \right]_{\theta=0}^{\theta=\pi}$$

with $a = z_p / R$ and $y = a^2 + 1 - 2a \cos \theta$.

$$dE_z = \frac{\rho \cdot dF}{2\varepsilon_0}, \text{ with } dF = \frac{1}{2} \left[ -\frac{1}{\sqrt{y}} + \frac{1}{a^2 \sqrt{y}} + \frac{\sqrt{y}}{a^2} \right]_{\theta=0}^{\theta=\pi} dR$$

Result for a hollow sphere:

$$dE_z = \frac{\rho \cdot R^2}{\varepsilon_0 z_p^2} dR$$

$$z_p > R_0: E_z = \int_0^{R_0} \frac{\rho \cdot R^2}{\varepsilon_0 z_p^2} dR = \frac{\rho \cdot R_0^3}{3\varepsilon_0 z_p^2}$$

$$z_p < R_0: E_z = \int_0^{z_p} \frac{\rho \cdot R^2}{\varepsilon_0 z_p^2} dR = \frac{\rho \cdot z_p^3}{3\varepsilon_0 z_p^2} = \frac{\rho \cdot z_p}{3\varepsilon_0}$$

(sphere shells $> z_p$ do not contribute !)
3. Solid sphere, homogeneously charged

Result for a solid sphere:

\[ E_z = \frac{\rho R^2}{\varepsilon_0 z_p^2} dR = \frac{\rho R^3}{3\varepsilon_0 z_p^2} \]

And with \( Q = \rho \cdot (4/3) \cdot \pi R_0^3 \):

outside: \( z_p > R_0 \):

\[ E_z = \frac{Q}{4\pi \varepsilon_0 z_p^2} \quad \text{inverse quadratic} \]

inside: \( z_p < R_0 \):

\[ E_z = \frac{Q \cdot z_p}{4\pi \varepsilon_0 R^3} \quad \text{linear} \]

These expressions are according to “Gauss”. 
$E$-field of a charged sphere

Contents:
1. A hollow sphere, homogeneously charged (conducting)
2. Idem., non-homogeneously charged
3. A solid sphere, homogeneously charged
4. Idem., non-homogeneously charged
4. Solid sphere, non-homogeneously charged

\[
E_z = \int_0^{R_0} dR \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\rho}{4\pi \varepsilon_0} \frac{R^2 \sin \theta [z_p - R \cos \theta]}{[\sin^2 \theta + (z_p - R \cos \theta)^2]^{3/2}}
\]

\[
E_{x,y} = \int_0^{R_0} dR \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\rho}{4\pi \varepsilon_0} \frac{R^2 \sin \theta \sin \theta \Phi(\phi)}{[\sin^2 \theta + (z_p - R \cos \theta)^2]^{3/2}}
\]

\[\rho = f(R, \theta, \phi) : X\text{- and } Y\text{-components present.}\]

In general:
Integration has to be performed numerically.
for homogeneous charge distribution:

\[ E = \frac{Q_{tot}}{4\pi\varepsilon_0 r^2} e_z \]

\[ E = 0 \]

Hollow: \( Q_{tot} = \sigma 4\pi R^2 \)

Solid: \( Q_{tot} = \rho 4\pi R^3 / 3 \)

total charge seems to be in center

E-field of a charged hollow or solid sphere