BioMedical Optics

Biomedical Optics

Light Scattering by Particles
Theoretical Aspects

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M.Sc. courses presented at
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Light Scattering by Particles
Theoretical Aspects

Contents: (movie part **I**)

1. Overview of scattering functions
2. Plots of various examples

Appendices: (movie part **II**)

A. Derivation of the dipole radiation formula
B. Derivation of the general scattering equation.

Literature: (most important)

H.C. van de Hulst: "Light scattering by small particles", 1957,1981, ISBN 0486642283, Dover Publ. New York.

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Contents: (movie part **I**)

1. Overview of scattering functions
   a) Huygens' principle of spherical waves
   b) General scattering expression
   c) Dipole (Rayleigh) scattering
   d) Rayleigh-Gans scattering
   e) Mie scattering
   f) Various other expressions

2. Plots of various examples
   Dipole, Rayleigh-Gans, Mie, etc.

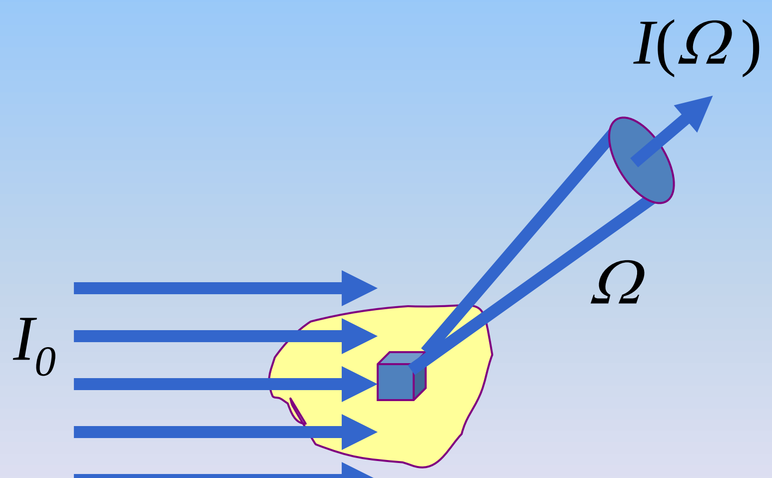
Definitions

Definitions:

Incident intensity: I_0

Scattered intensity: $I\ (\Omega)$

Ω = **solid scattering angle**



Scattering cross section σ
per particle [m^2]

= apparent effective "shadow area" for scattering

σ may be dependent on the angle of scattering

$\frac{d\sigma}{d\Omega}$ **differential scattering cross section**

$$I(\Omega) = \frac{d\sigma}{d\Omega} I_0$$

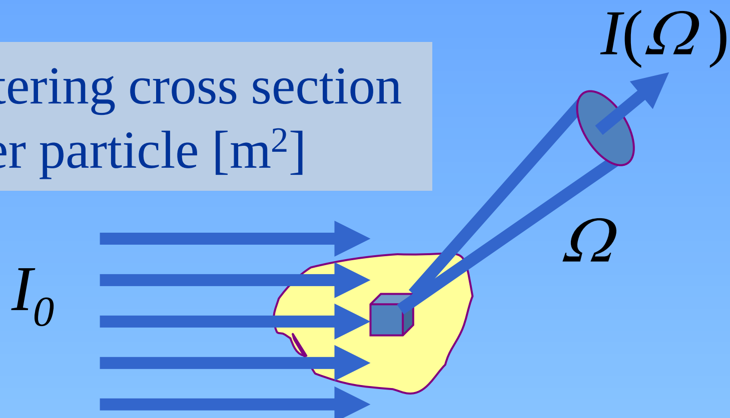
$$\left[\frac{W}{sr}\right] = \left[\frac{m^2}{sr}\right]\left[\frac{W}{m^2}\right]$$

$$\sigma_{tot} = \iint_{\Omega} \left(\frac{\partial \sigma}{\partial \Omega}\right) d\Omega$$

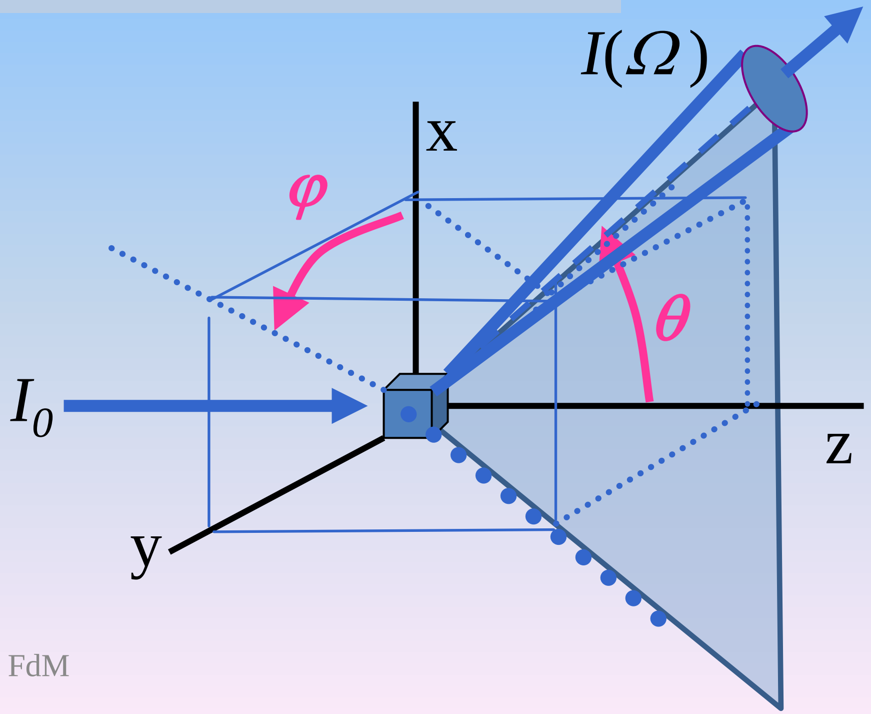
total scattering cross section

Definitions

Scattering cross section σ per particle [m^2]



Incident beam // +Z-axis



$$I(\Omega) = \frac{d\sigma}{d\Omega} I_0$$

Scattering angles:

θ = polar angle (from Z-axis)

φ = azimuthal angle (in XY-plane)

Scattering function:

$$p(\theta,\varphi) = \frac{\partial\sigma}{\partial\Omega}$$

$$I(\Omega) = p(\theta,\varphi) I_0$$

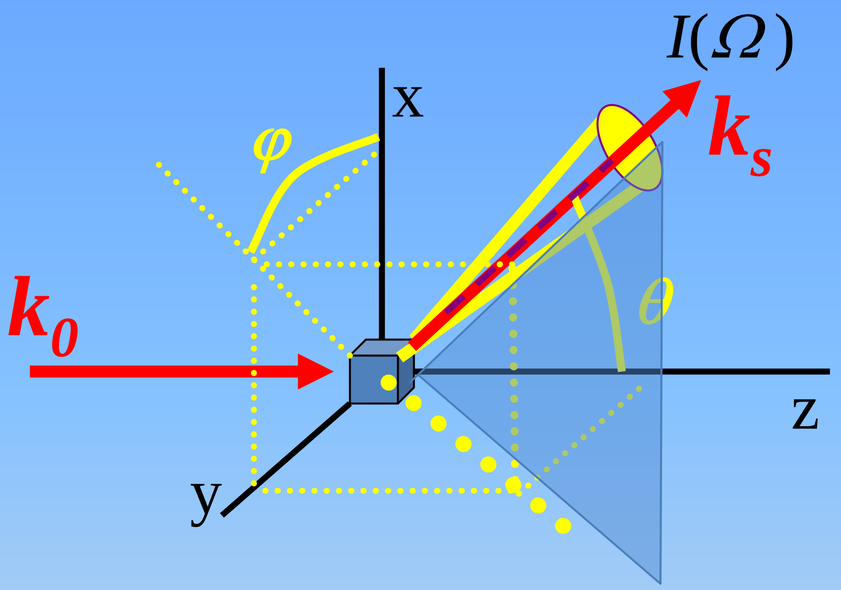
Normalization:

$$\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} p(\theta,\varphi).\sin\theta.d\theta d\varphi = 1$$

Definitions

Wave vectors: $\boldsymbol{k_0}$ and $\boldsymbol{k_s}$ (or $\boldsymbol{k}$)

$$k = \frac{2\pi}{\lambda} \quad ; \quad \lambda_{medium} = \frac{\lambda_{vacuum}}{n}$$



Scattering / Absorption coefficients: $\boldsymbol{\mu}$ [m^{-1}]

$\boldsymbol{\mu} = \boldsymbol{n}\,\boldsymbol{\sigma}$

[m^{-1}] = [m^{-3}] [m^{2}]

$\boldsymbol{n}$ = nr. of particles per m^3

Beam attenuation coefficients:

- scattering: μ_s [m^{-1}]
- absorption: μ_a
- total: $\mu_t = \mu_s + \mu_a$

Lambert-Beer Law:

$$I(z) = I_0 . exp(-\mu\, z)$$

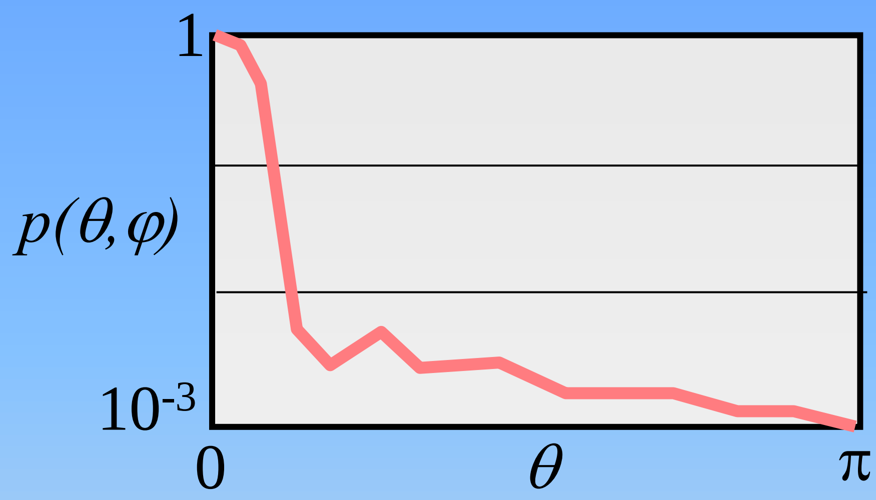
Mean-free-path: $MFP = \mu_t^{\ -1}$ [m]

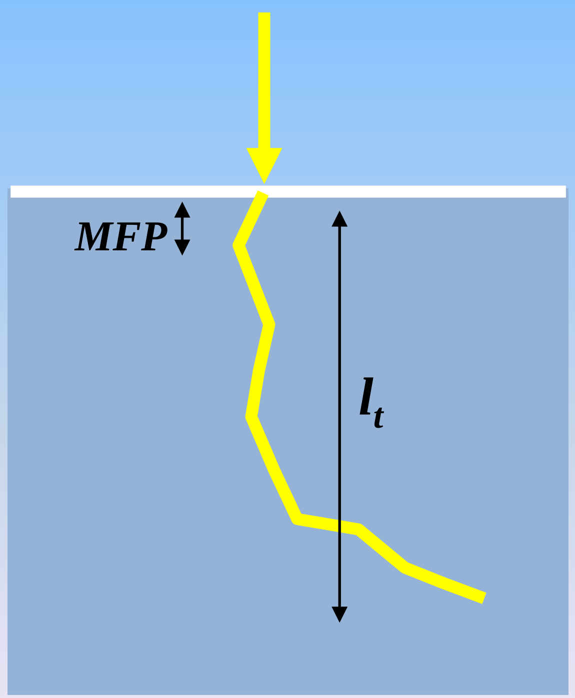
Albedo : μ_s / μ_t [-]

Definitions

Scattering in tissue is predominantly forward:

$g = \langle \cos\theta \rangle \quad > \approx 0.9$





Reduced scattering coefficient:

$\mu_s' = (1 - g)\,\mu_s \qquad << \mu_s$

Transport mean-free-path:

$l_t = 1 / [\mu_s' + \mu_a] \;\; >> MFP$

$MFP = 1 / [\mu_s + \mu_a]$

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Overview

Overview of theories:

	Basic:
EM	Electromagnetic Theory from Maxwell's Equations
D	Dipolar (Rayleigh) scattering
RG	Rayleigh-Gans scattering
M	Mie scattering
	Other Scattering Functions:
HG	Henyey-Greenstein
I	Isotropic
PF	Peaked-forward
	Advanced Models:
G	Groenhuis
B	Bonner
P	Patterson
F	Farrell
	Fundamental: (not in this presentation) Transport Equation and Diffusion Scattering

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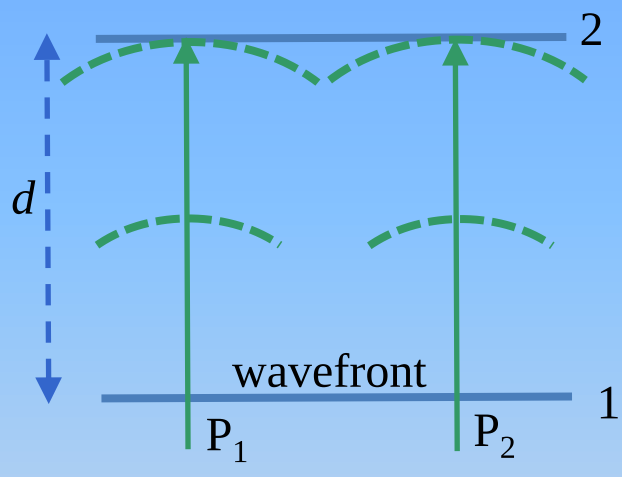
Basic Theories: Principle of Huygens

Field propagation by spherical waves, emitted by all points of the wavefront



$\lambda << d$ $k = 2\pi/\lambda$

Fresnel (1818) derived:

"**Disturbance**"
from any point of plane 1 to any point of plane 2 (at mutual distance r):

~ spherical wave Φ :

$$\Phi = \frac{e^{-ikr}}{r}$$

Integrate over plane 2:

Result: "Disturbance" E caused by a "disturbance" E_0 , present at a wavefront area element dS, in a point at a distance r from dS :

$$E = \frac{i}{r\lambda} e^{-ikr} dS . E_0$$

Factor $\boldsymbol{i}$ (= $\boldsymbol{e}^{\,i\pi/2}$) causes phase shift ½ π in scattered wave (cos ↔ sin).

This is the basic formula for all light scattering theories.

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Basic EM Scattering Theory (1)

EM D RG M
HG I PF
G B P F

Electromagnetic Theory from Maxwell's Equations $\rho = 0 \; ; j = 0$

$$\nabla \bullet \vec{\boldsymbol{D}} = 0 \quad ; \quad \nabla \bullet \vec{\boldsymbol{B}} = 0 \quad ; \quad \nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \quad ; \quad \nabla \times \vec{\boldsymbol{B}} = \mu_0 \frac{\partial \vec{\boldsymbol{D}}}{\partial t}$$

Wave Equation:

$$\nabla \times \nabla \times \vec{\boldsymbol{E}} = -\mu_0 \frac{\partial^2 \vec{\boldsymbol{D}}}{\partial \boldsymbol{t}^2}$$

Insert material properties:
dielectric constant
$\boldsymbol{\varepsilon_1 = \varepsilon_0 \,.\, (\varepsilon_r - 1)} = f(\boldsymbol{r}, t)$

$$\vec{\boldsymbol{D}}(\vec{\boldsymbol{r}},t) = \varepsilon(\vec{\boldsymbol{r}},t).\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},t)$$

$$\varepsilon(\vec{\boldsymbol{r}},t) = \varepsilon_0(\vec{\boldsymbol{r}},t) + \varepsilon_1(\vec{\boldsymbol{r}},t) + \ldots$$

Fields: incident: $\boldsymbol{E_0}$
scattered: $\boldsymbol{E_1}$
total: $\boldsymbol{E = E_0 + E_1}$

$$\vec{\boldsymbol{E}}_{\boldsymbol{0}}(\boldsymbol{r},t) = \vec{\boldsymbol{E}}_{\boldsymbol{0}m}\, exp\left[i(\vec{\boldsymbol{k}}_0 \bullet \vec{\boldsymbol{r}} - \omega_0 t)\right]$$

$k = 2\pi/\lambda \quad ; \quad \omega = 2\pi f$

Assume: $\varepsilon_1 << \varepsilon_0$ and $\boldsymbol{E_1 << E_0}$; 1st order approximation

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Basic EM Scattering Theory (2)

Wave Equation: $$\nabla \times \nabla \times \vec{\boldsymbol{E}} = -\mu_0 \frac{\partial^2 \vec{\boldsymbol{D}}}{\partial \boldsymbol{t}^2} \quad ; \quad \vec{\boldsymbol{D}} = \varepsilon \, \vec{\boldsymbol{E}}$$

Assume: $\varepsilon_1 << \varepsilon_0$ and $\boldsymbol{E_1} << \boldsymbol{E_0}$; 1st order approximation

Insert: $\boldsymbol{E_0}+\boldsymbol{E_1}$ for $\boldsymbol{E}$, and of $\varepsilon_0+\varepsilon_1$ for ε in the Wave Equation

Use: $\nabla \times \nabla \times \vec{\boldsymbol{E}} = -\nabla^2 \vec{\boldsymbol{E}} + \nabla(\nabla \bullet \vec{\boldsymbol{E}}) = -\nabla^2 \vec{\boldsymbol{E}}$ since $\nabla \bullet \vec{\boldsymbol{E}} = 0$

0th order: $$\nabla^2 \vec{\boldsymbol{E}}_{\boldsymbol{0}} = \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{\boldsymbol{E}}_{\boldsymbol{0}}$$

=> Incoming field $\boldsymbol{E_0}$

1st order: $$\nabla^2 \vec{\boldsymbol{E}}_1 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{\boldsymbol{E}}_1 = \mu_0 \frac{\partial^2}{\partial t^2} (\varepsilon_1 \vec{\boldsymbol{E}}_0) - \frac{1}{\varepsilon_0} \nabla(\nabla \bullet (\varepsilon_1 \vec{\boldsymbol{E}}_0))$$

so: $$f_1(\vec{\boldsymbol{E}}_{\boldsymbol{1}}) = f_0(\varepsilon_1 \vec{\boldsymbol{E}}_{\boldsymbol{0}})$$

Problem: Find $\boldsymbol{E_1} = f(\varepsilon_1 \, \boldsymbol{E_0})$

Basic EM Scattering Theory (3)

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EM D RG M
HG I PF
G B P F

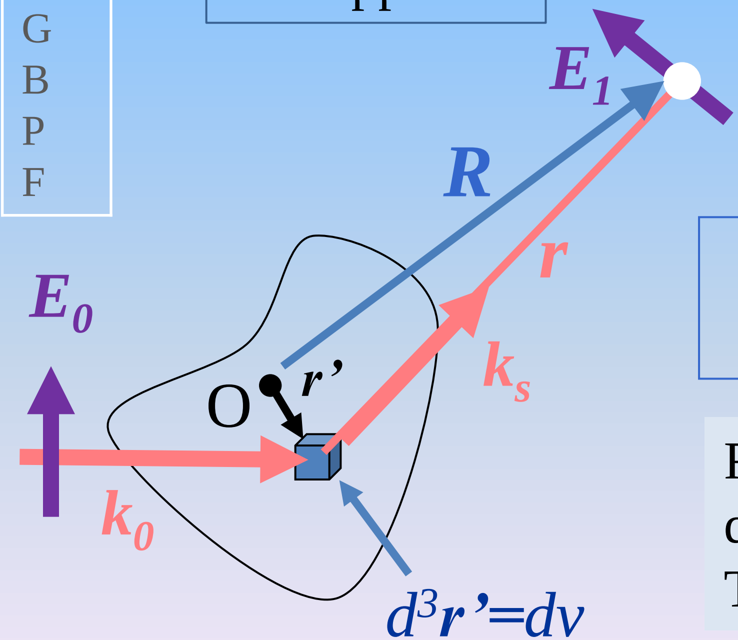
Problem: Find $E_1 = f(\varepsilon_1\ E_0)$

Solution:

$$\vec{E}_1(\vec{R},t) = \frac{1}{4\pi\varepsilon_0 R}\iiint_{vol} d^3\vec{r}'.\vec{k}_s \times \vec{k}_s \times \left\{\varepsilon_1(\vec{r}',t').\ \vec{E}_0(\vec{r}',t')\right\}$$

- $1/R$: Spherical outgoing wave
- Volume integration
- Polarization direction
- ε : Material properties
- E_0: Incident field vector

Derivation: See Appendix



Retarded time: $t' = t - (r - r')/c$
due to flight time towards detector.
This retarded time accounts for phase differences

$$\exp[-i(\vec{k}_s - \vec{k}_0)\bullet(\vec{R}-\vec{r}') - i\omega_0 t]$$

Basic EM Scattering Theory (4)

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EM
D
RG
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Determine direction of Polarization:

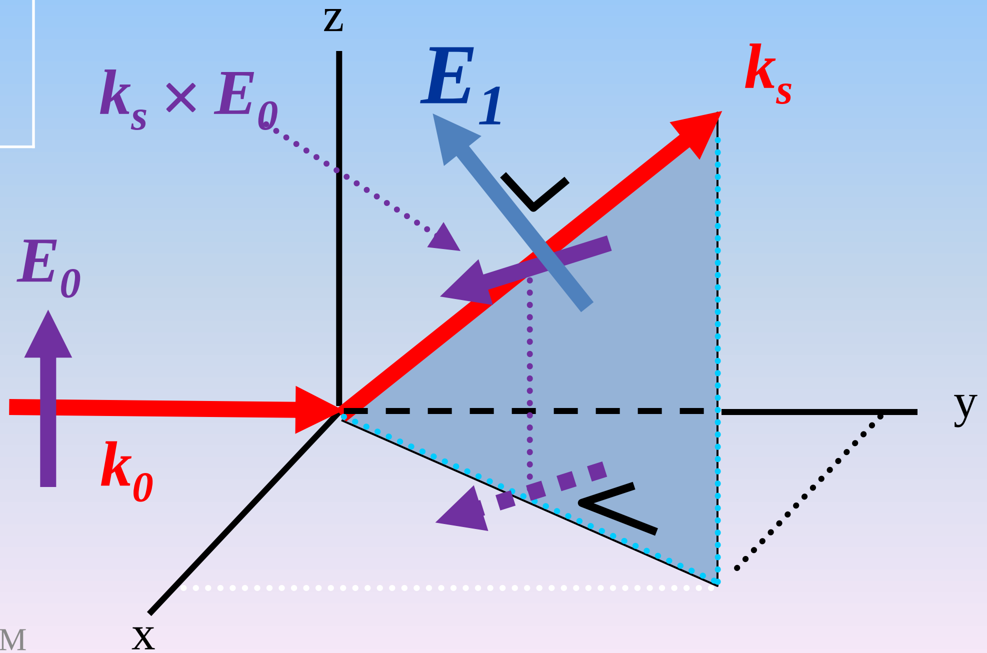
Here $\varepsilon(\vec{r}',t')$ is scalar function

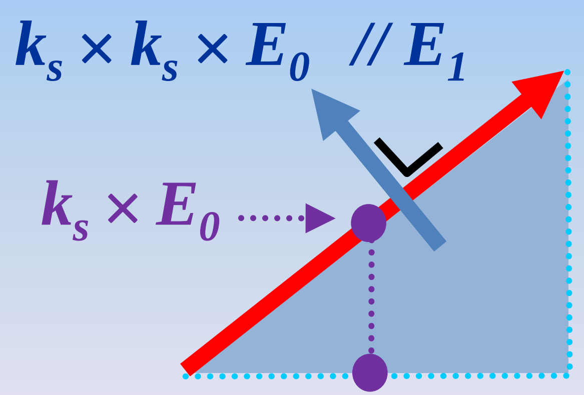
Solution:

$$\vec{\boldsymbol{E}}_1(\vec{\boldsymbol{r}},t) = \frac{1}{4\pi\varepsilon_0 r}\iiint_{vol} d^3\vec{\boldsymbol{r}}'.\vec{\boldsymbol{k}}_s \times \vec{\boldsymbol{k}}_s \times \left\{ \varepsilon_1(\vec{\boldsymbol{r}}',t').\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{r}}',t') \right\}$$

$$\boldsymbol{E}_1 \,//\, \left[\boldsymbol{k}_s \times \boldsymbol{k}_s \times \boldsymbol{E}_0\right]$$

$\boldsymbol{E}_0$: vertical polarization





$\boldsymbol{k}_s \times \boldsymbol{E}_0$ pointing out-of-plane

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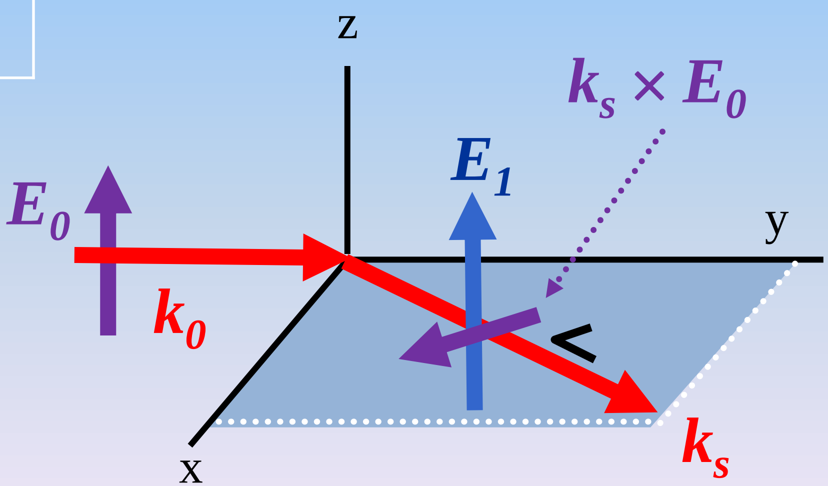
Basic EM Scattering Theory (5)

Determine direction of Polarization

Here $\varepsilon(\vec{r}\,',t')$ is scalar function

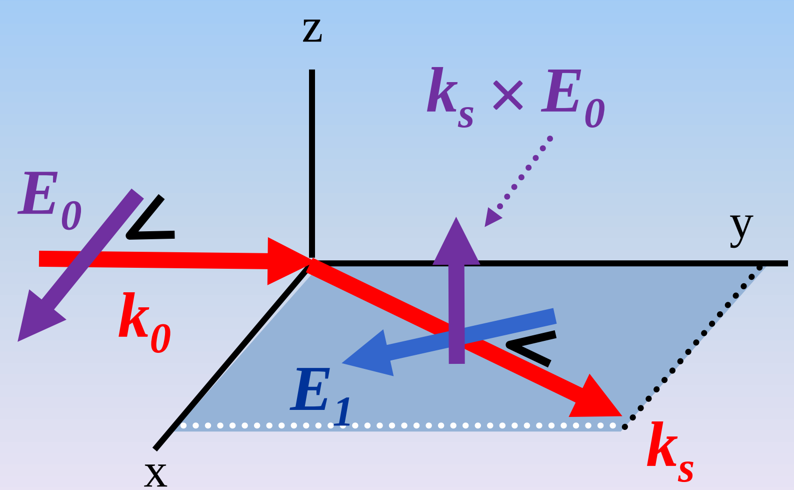
Solution: $$\vec{E}_1(\vec{r},t) = \frac{1}{4\pi\varepsilon_0 r}\iiint_{vol} d^3\vec{r}\,'.\vec{k}_s\times\vec{k}_s\times\left\{\varepsilon_1(\vec{r}\,',t').\vec{E}_0(\vec{r}\,',t')\right\}$$

In-plane scattering:
vertical polarization



No depolarization

In-plane scattering:
horizontal polarization



Depolarization present

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Basic EM Scattering Theory (6)

EM
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HG
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Solution: Electric Field Strength:

Here $\varepsilon(\vec{r}',t')$ is scalar function

$$\vec{\boldsymbol{E}}_1(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\varepsilon_0 r}\iiint_{vol} d^3\vec{\boldsymbol{r}}'.\vec{\boldsymbol{k}}_s\times\vec{\boldsymbol{k}}_s\times\left\{\varepsilon_1(\vec{\boldsymbol{r}}',t').\vec{\boldsymbol{E}}_0(\vec{\boldsymbol{r}}',t')\right\}$$

Intensity = autocorrelation function of Field Strength:

$$I(\vec{\boldsymbol{r}},t)\propto<\vec{\boldsymbol{E}}_1^{\,*}(\vec{\boldsymbol{r}},0)\bullet\vec{\boldsymbol{E}}_1(\vec{\boldsymbol{r}},t)>\equiv$$

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{+T}d\tau\,.\vec{\boldsymbol{E}}_1^{\,*}(\vec{\boldsymbol{r}},\tau)\bullet\vec{\boldsymbol{E}}_1(\vec{\boldsymbol{r}},\tau+t)$$

Intensity ~ (Field Strength) ^{2}

$\sim k^{4} \sim \lambda^{-4}$

$\sim r^{-2}$

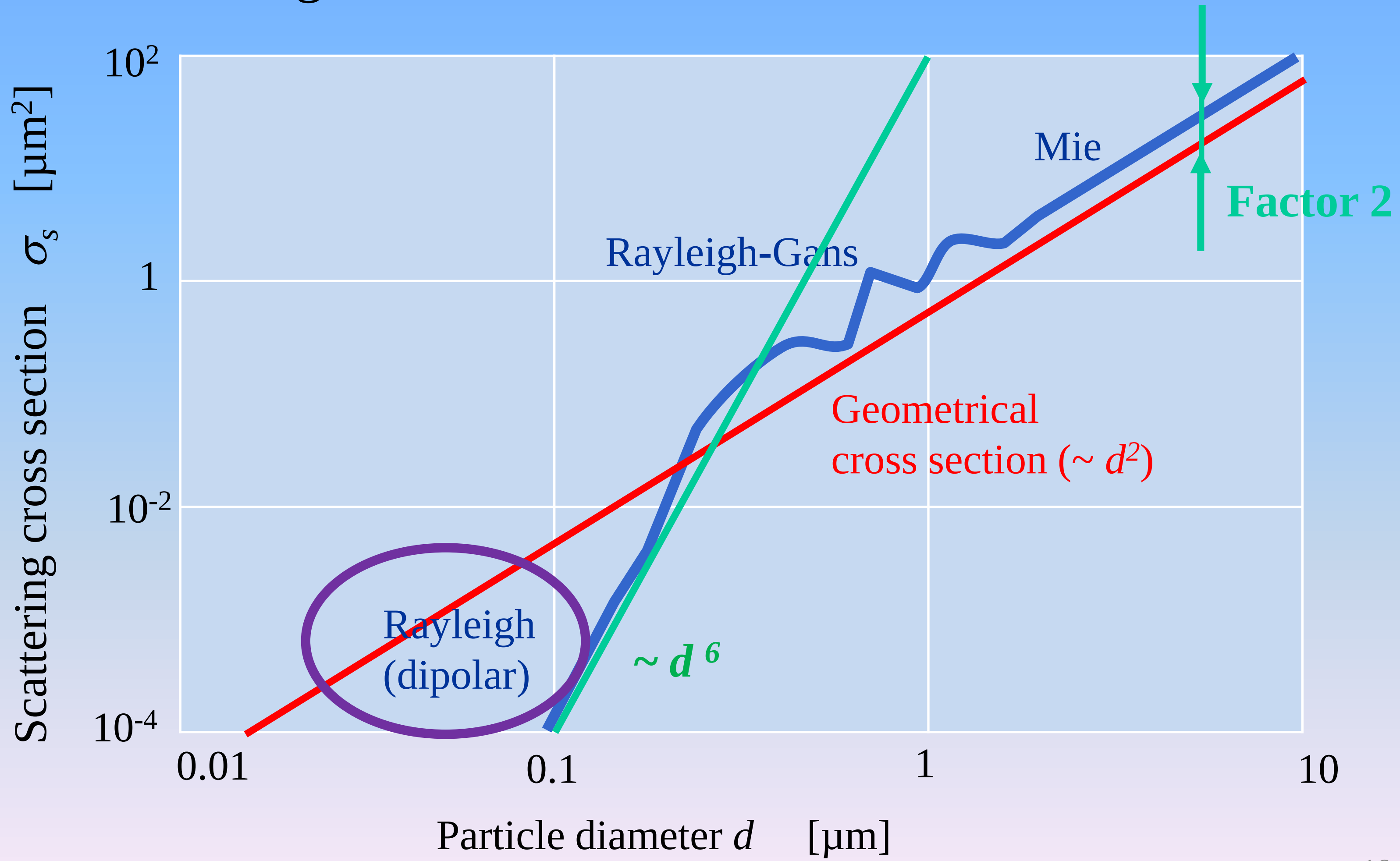
$\sim \varepsilon_1^{2}$

~ volume^2

NB. If ε = tensor function of coordinates,
then polarization direction will differ.

Optical properties: scattering

Scattering cross section, for λ = 500 nm and n = 1.5



Dipole (Rayleigh) Scattering (1)

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EM
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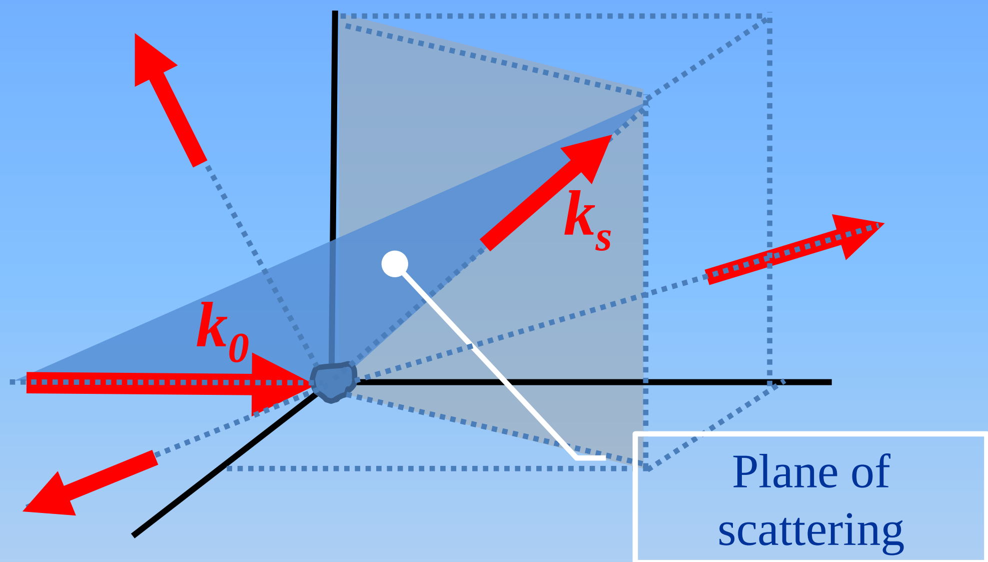


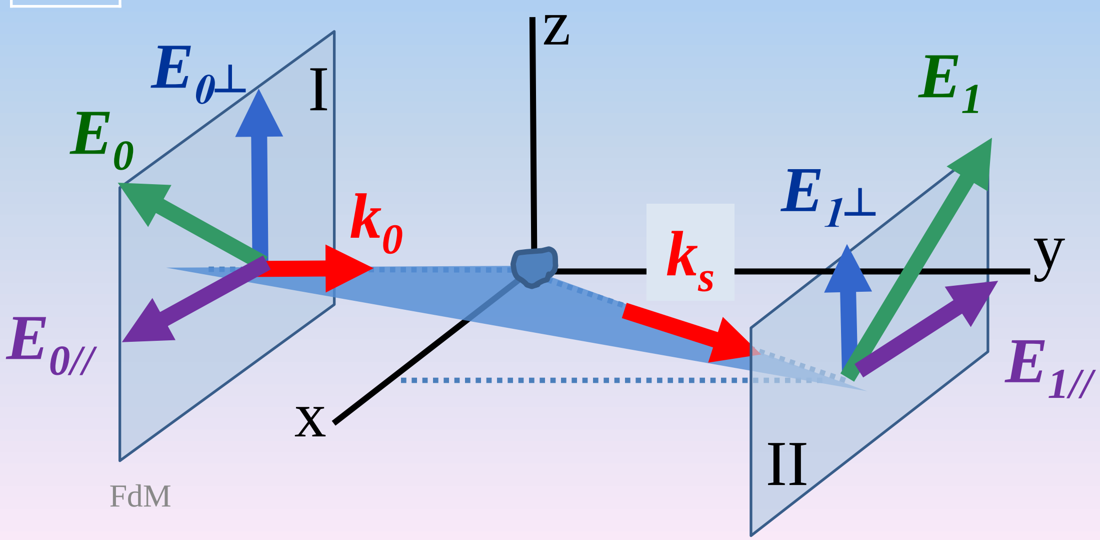
Wavevectors:

- Incident field: $\boldsymbol{k_0}$
- Scattered field: $\boldsymbol{k_s}$

Define **plane of scattering** using $\boldsymbol{k_0}$ and $\boldsymbol{k_s}$

Define **coordinate axes:** x, y, z

Plane I , II $\perp$ $\boldsymbol{k_0}$, $\boldsymbol{k_s}$ resp.



Electric field:

- Incident: $\boldsymbol{E_0}$
- Scattered: $\mathbf{E_1}$ (or $\boldsymbol{E_s}$)

$\boldsymbol{E}$-field has components $\boldsymbol{E_\perp}$ and $\boldsymbol{E_{//}}$ w.r.t. this scattering plane.

Dipole (Rayleigh) Scattering (2)

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EM **D** RG M
HG I PF
G B P F

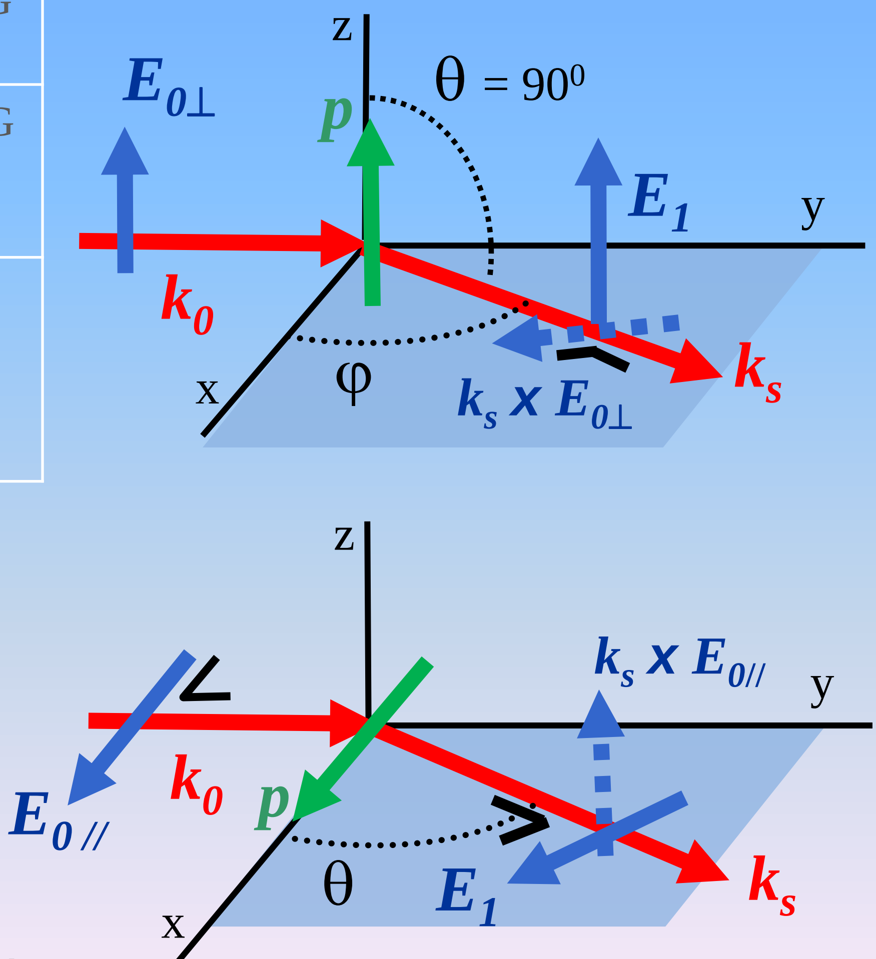
$$\boldsymbol{E}_1 \,//\, \left[\boldsymbol{k}_s \times \boldsymbol{k}_s \times \boldsymbol{E}_0\right]$$

Dipole $\boldsymbol{p}$ induced by E-field $\boldsymbol{E_0}$

$\boldsymbol{p} = \alpha\, \boldsymbol{E_0}$ $\quad\quad$ $\boldsymbol{E_1} \perp \boldsymbol{k_s}$

α ~ relative dielectric constant

$\boldsymbol{E}$-field has components $\boldsymbol{E}_\perp$ and $\boldsymbol{E}_{//}$ w.r.t. the scattering plane.

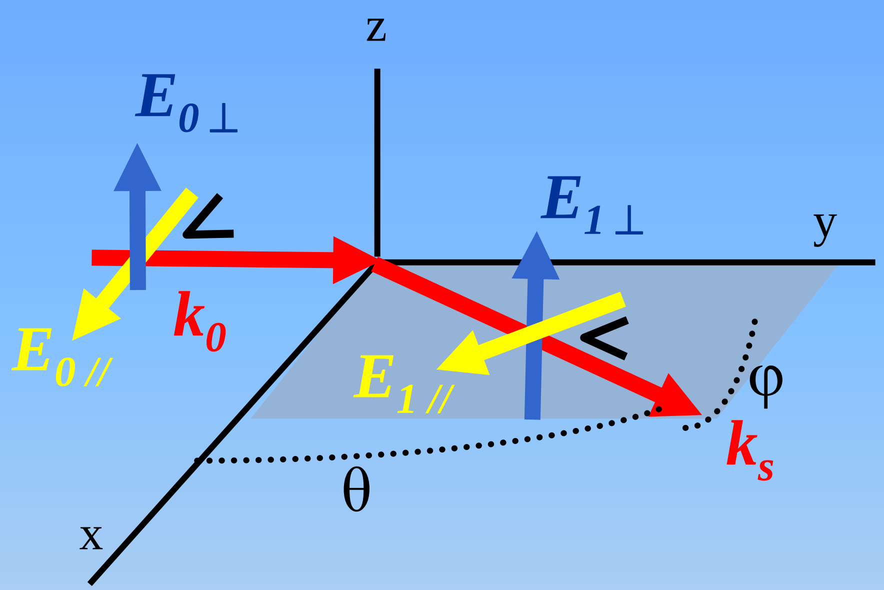


$$\begin{bmatrix} \boldsymbol{E}_{1\perp} \\ \boldsymbol{E}_{1//} \end{bmatrix} \sim \frac{\alpha\, k_s^{\,2}}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 \\ \sin\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_{0\perp} \\ \boldsymbol{E}_{0//} \end{bmatrix}$$

$$\begin{bmatrix} I_\perp \\ I_{//} \end{bmatrix} \sim \frac{\alpha^2 k_s^{\,4}}{r^2} \begin{bmatrix} 1 \\ \sin^2\theta \end{bmatrix} \begin{bmatrix} E_{0\perp}^{\;2} \\ E_{0//}^{\;2} \end{bmatrix}$$

Derivation in Appendix

Dipole (Rayleigh) Scattering (3)



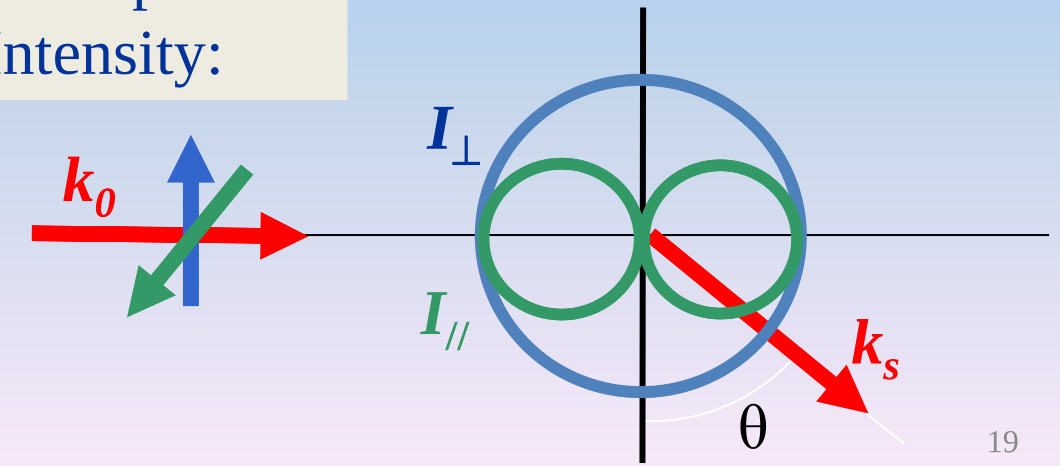
$$\begin{bmatrix} E_{1\perp} \\ E_{1//} \end{bmatrix} \sim \frac{\alpha k_s^{\,2}}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 & \\ & \sin\theta \end{bmatrix} \begin{bmatrix} E_{0\perp} \\ E_{0//} \end{bmatrix}$$

Lorentz:
Spheres (radius a ; volume V):

$$\alpha = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}\,\frac{3V}{4\pi} = \frac{\varepsilon_r - 1}{\varepsilon_r + 2}\,a^3$$

$\varepsilon_r = n^2$
ε_r = relative dielectric constant
n = refractive index.

Polar plot of Intensity:



NB. In stead of θ, angle φ is used,
$\varphi = 90^0 - \theta$
$\rightarrow \sin\theta = \cos\varphi$

Dipole (Rayleigh) Scattering (4)

Dipolar Scattering for 1 scatterer:

$$\begin{bmatrix} \boldsymbol{E}_{1\perp} \\ \boldsymbol{E}_{1//} \end{bmatrix} \propto \frac{\alpha\, k_s^{\,2}}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 \\ \sin\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_{0\perp} \\ \boldsymbol{E}_{0//} \end{bmatrix}$$

If n scatterers per m^3 (each radius a):

$$\begin{bmatrix} \boldsymbol{E}_{1\perp} \\ \boldsymbol{E}_{1//} \end{bmatrix} \propto nV \frac{\alpha\, k_s^{\,2}}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 \\ \sin\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_{0\perp} \\ \boldsymbol{E}_{0//} \end{bmatrix} \text{; with } \quad V = \tfrac{4}{3}\pi a^3$$

$$\begin{bmatrix} I_{\perp} \\ I_{//} \end{bmatrix} \propto n^2 a^6 \frac{\alpha^2 k_s^{\,4}}{r^2} \begin{bmatrix} 1 \\ \sin^2\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_{0\perp}^{\,2} \\ \boldsymbol{E}_{0//}^{\,2} \end{bmatrix}$$

"Natural" light :
(randomly polarized)
$I_{nat} = ½\,[I_{//} + I_{\perp}] \sim ½\,[\,1+\sin^2\theta]$

Intensity is proportional to:

- $\boldsymbol{\alpha}^{\boldsymbol{2}}$; α = dipole response on $\boldsymbol{E_0}$-field: $\boldsymbol{p} = \alpha \boldsymbol{E_0}$
- $\boldsymbol{a}^{\boldsymbol{6}}$; a = particle radius
- $\boldsymbol{V}^{\boldsymbol{2}}$; V = volume
- $\boldsymbol{k_s}^{\boldsymbol{4}} \sim \boldsymbol{\lambda}^{\boldsymbol{-4}}$; λ = wavelength
- $\boldsymbol{r}^{\boldsymbol{-2}}$; r = detector distance (spherical wave)

Dipole (Rayleigh) Scattering (5)

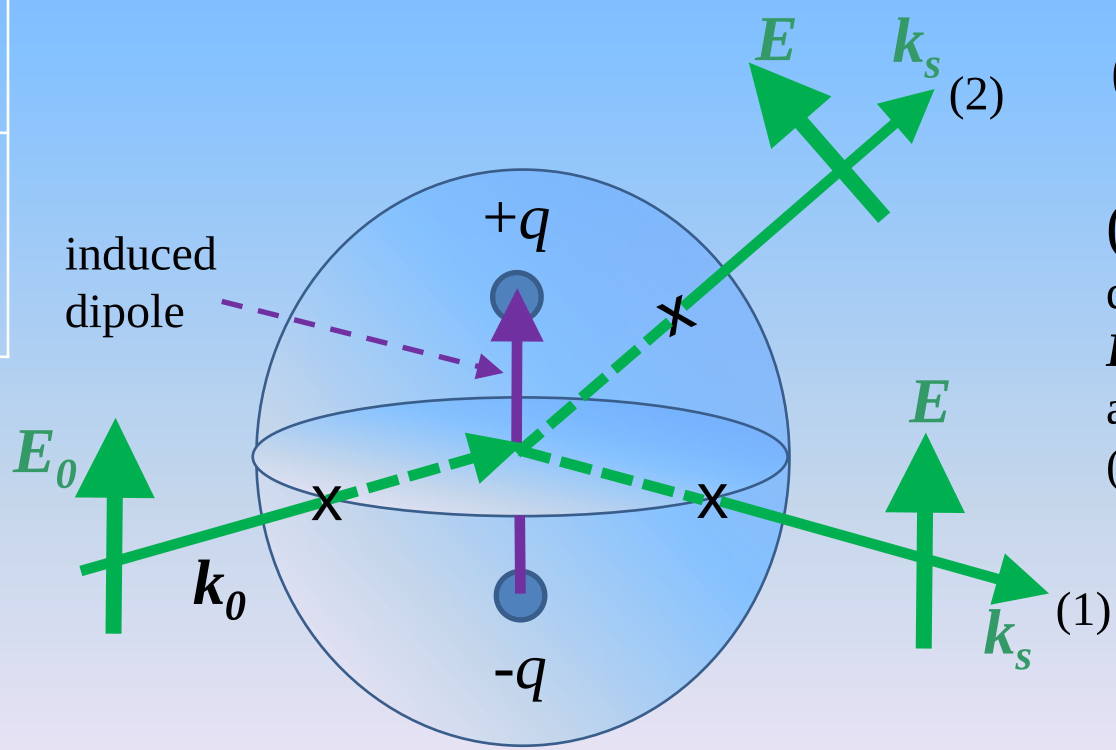
EM
D
RG
M

HG
I
PF

G
B
P
F

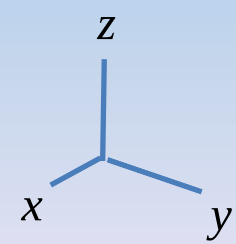
Dipolar Scattering for 1 scatterer:
n scatterers / m^3

$$\begin{bmatrix} E_{1\perp} \\ E_{1//} \end{bmatrix} \propto nV \frac{\alpha\, k_s^{\,2}}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 \\ \sin\theta \end{bmatrix} \begin{bmatrix} E_{0\perp} \\ E_{0//} \end{bmatrix} \text{; with } V = \tfrac{4}{3}\pi a^3$$



(1): $\theta = 90^0$: ***E*** **// E$_0$:** no depolarization

(2): $\theta \neq 90^0$:
depolarization present,
E has *z*-component ⊥ *xy*-plane
and component // *xy*-plane
(~ sin φ).



Dipole (Rayleigh) Scattering (6)

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EM
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RG
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HG
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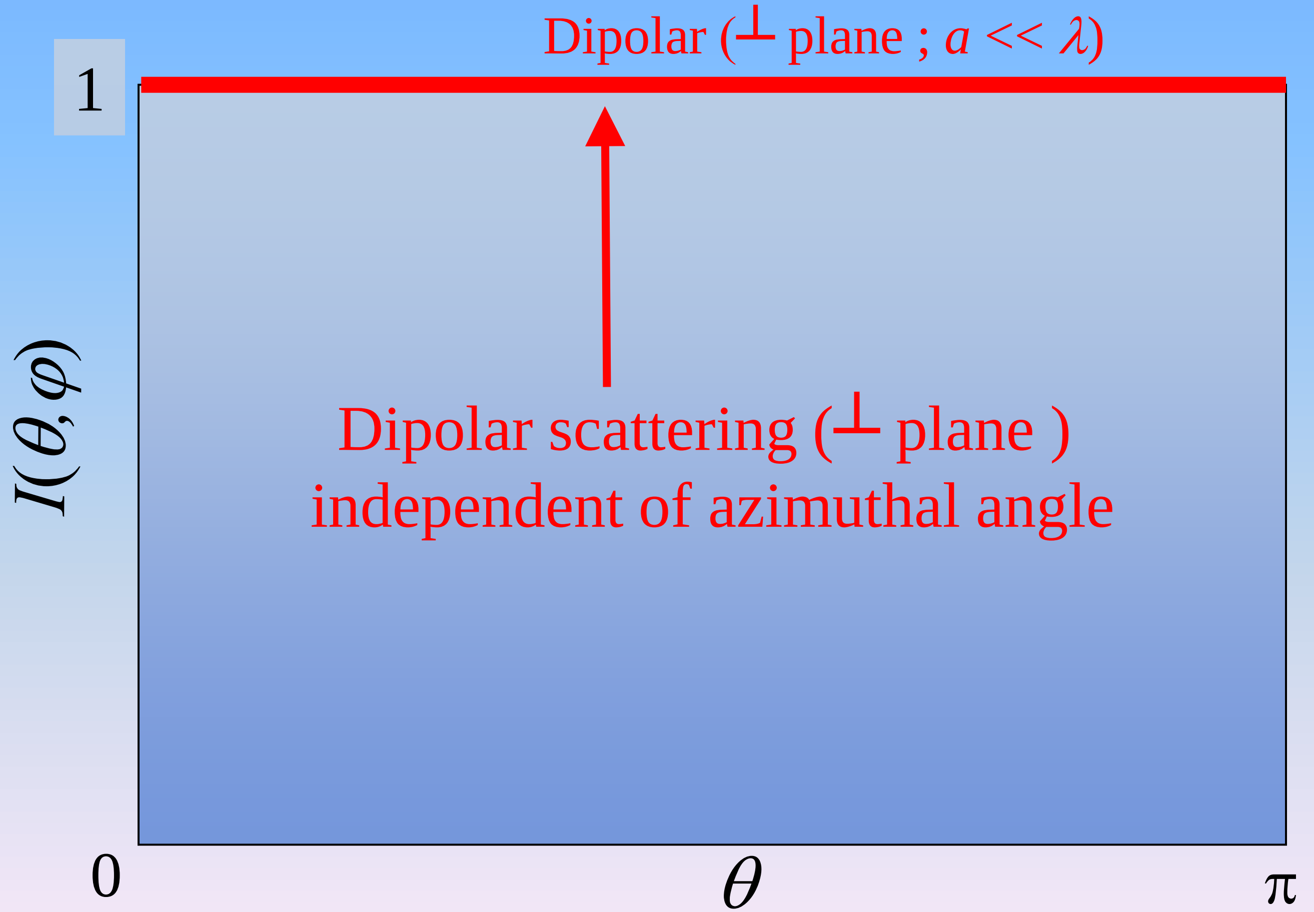
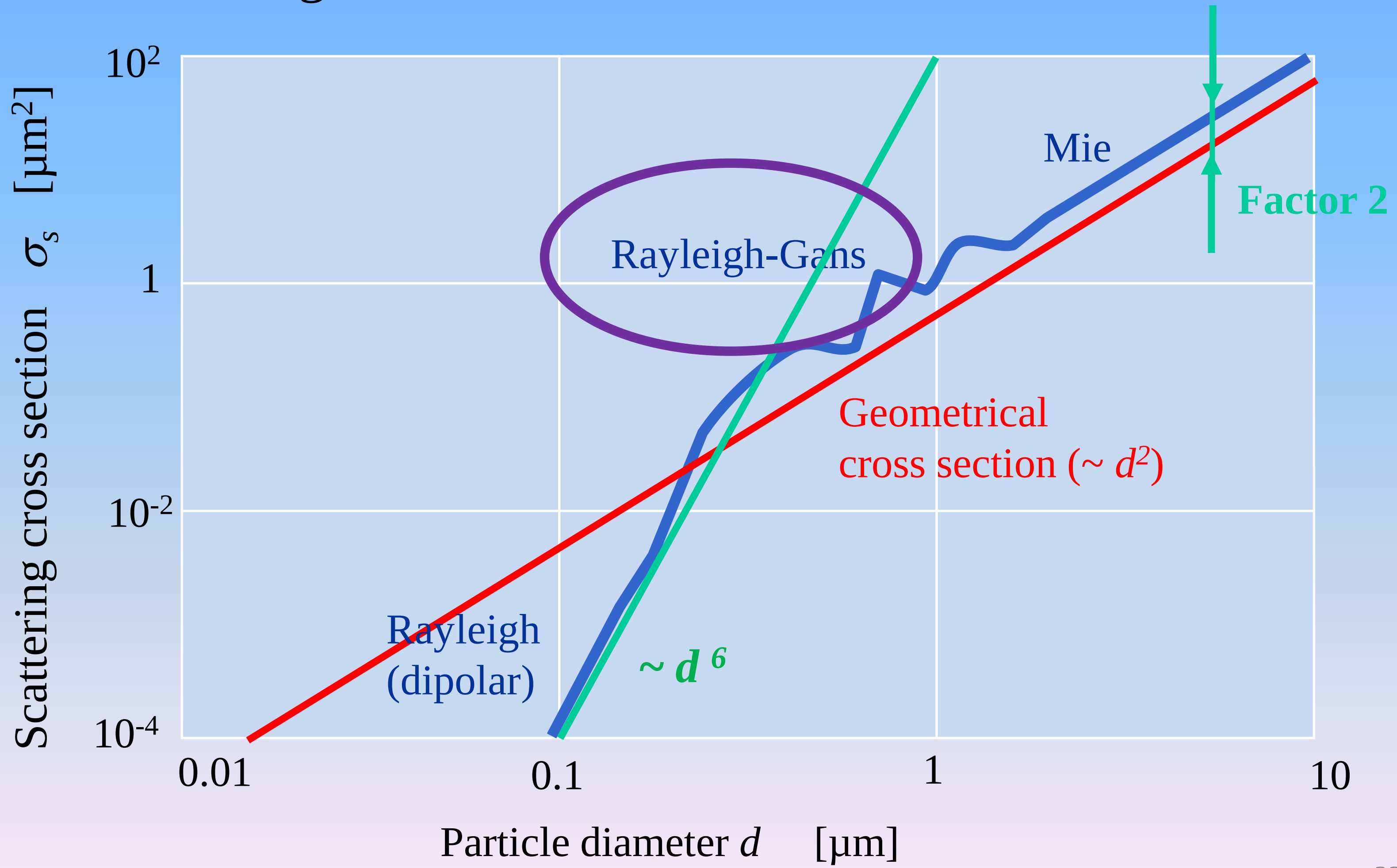
In-plane Scattering Intensity $I(\theta,\varphi)$
(vertical polarization)

a = particle radius



Optical properties: scattering

Scattering cross section, for λ = 500 nm and n = 1.5



Rayleigh-Gans scattering (1)

EM
D
RG
M

HG
I
PF

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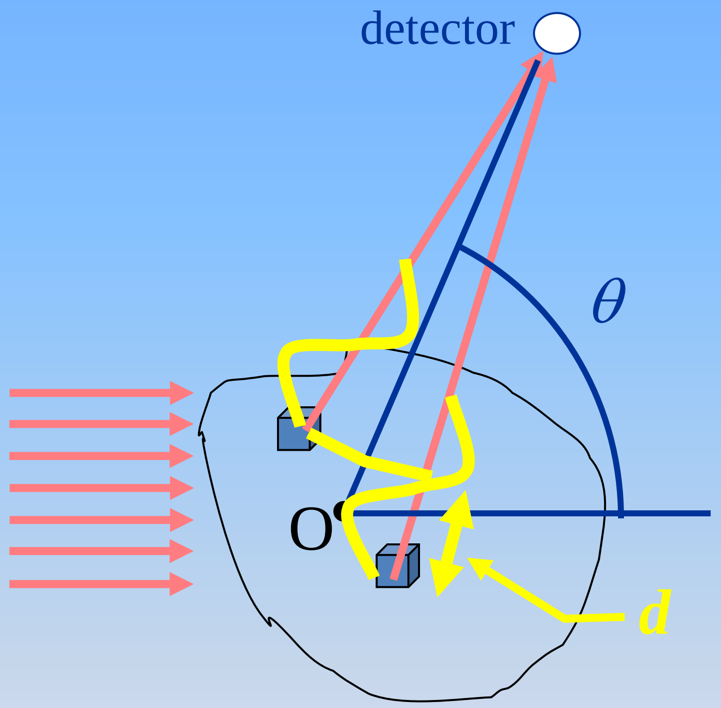
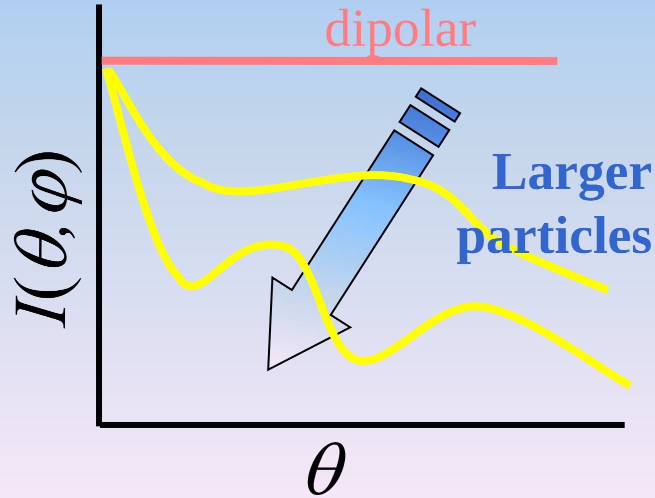
Particles with radius a, **not** << λ : ***Rayleigh-Gans***



Scattering from different parts may have different phases and will interfere at detector.

Destructive interference possible (*e.g.* if $d = ½ \lambda$) dependent upon angle θ and upon particle shape.



Rayleigh-Gans is approximative treatment

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Rayleigh-Gans scattering (2)

EM
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RG
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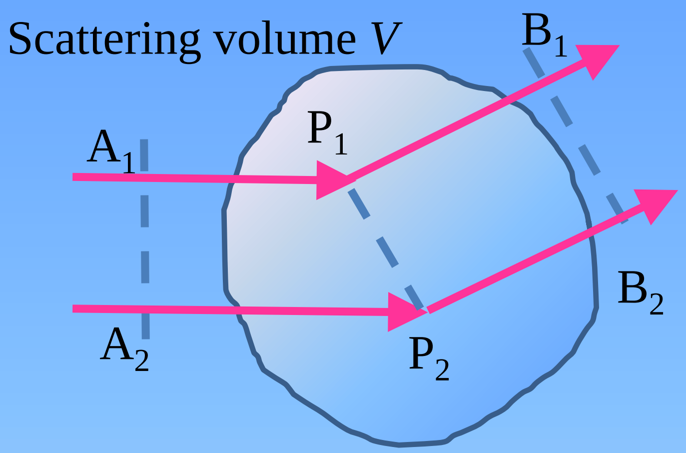
Rayleigh-Gans scattering:

accounts for phase differences when particles grow larger:



If n scatterers per m^3 :

$$\begin{bmatrix} dE_{1\perp} \\ dE_{1//} \end{bmatrix} \propto n \frac{\alpha\, k_s^{\,2}}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 \\ \sin\theta \end{bmatrix} \begin{bmatrix} E_{0\perp} \\ E_{0//} \end{bmatrix} e^{i\partial}\, dV$$

∂ accounts for phase differences;

$$\begin{bmatrix} E_{1\perp} \\ E_{1//} \end{bmatrix} \propto nV \frac{\alpha\, k_s^{\,2}}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 \\ \sin\theta \end{bmatrix} \begin{bmatrix} E_{0\perp} \\ E_{0//} \end{bmatrix} .R(\theta,\varphi)$$

with $$R(\theta,\varphi) = \frac{1}{V} \iiint_V e^{i\partial}\, dV$$

Final result depends on R-function.
e.g. Phase difference between 2 wave fronts: A and B :

$$\varphi = 2\pi \frac{(B_1P_1 + P_1A_1) - (B_2P_2 + P_2A_2)}{\lambda} \; ; \; \lambda = \frac{2\pi}{k_s}$$

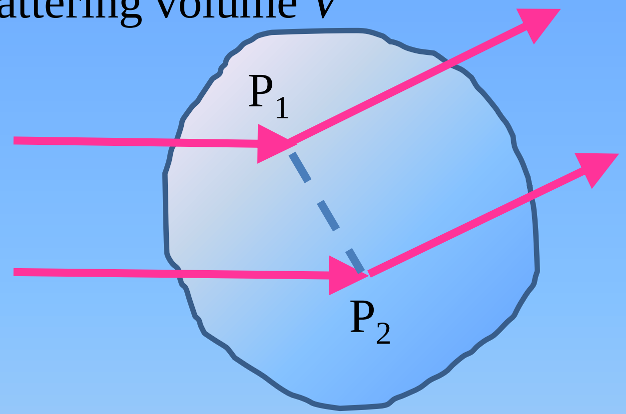
Result for "natural" incident light: $I_{nat} = ½\,[I_{//} + I_{\perp}]$; a = particle radius:

$$I_{nat} \propto n^2 V^2 \frac{\alpha^2 k_s^{\,4}}{(4\pi\varepsilon_0 r)^2} \frac{1+\sin^2\theta}{2} E_0^{\,2} \left|R(\theta,\varphi)\right|^2 \quad ; \quad V^2 \propto a^6$$

Rayleigh-Gans scattering (3)

Rayleigh-Gans scattering:

accounts for phase differences between paths from different spots in the scattering medium, when particles grow larger



Phase function:

$$R(\theta,\varphi) = \frac{1}{V}\iiint_V e^{i\partial}\,dV$$

Examples:

1. solid homogeneous sphere:
   (J = Bessel function)

$$R(\theta,\varphi) = \sqrt{\frac{9\pi}{2u^3}}.J_{3/2}(u) \quad ; \quad u = 2ka.\sin(\tfrac{1}{2}\theta)$$

2. inhomogeneous sphere : $\varepsilon = f(r)$:

$$R(\theta,\varphi) = \frac{1}{\varepsilon_c}\int_0^\infty 4\pi r^2 \varepsilon(r)\frac{\sin v}{v}dr \; ; \; v = 2kr.\sin(\tfrac{1}{2}\theta) \; ; \; \varepsilon_c = \int_0^\infty 4\pi r^2 \varepsilon(r)dr$$

3. cylinder (circular, length l, radius a); disk and rod, and other shapes:
   see Van de Hulst: "Light scattering by small particles"

4. for (tissue) particles with random orientations: use 1 or 2 as approximation .

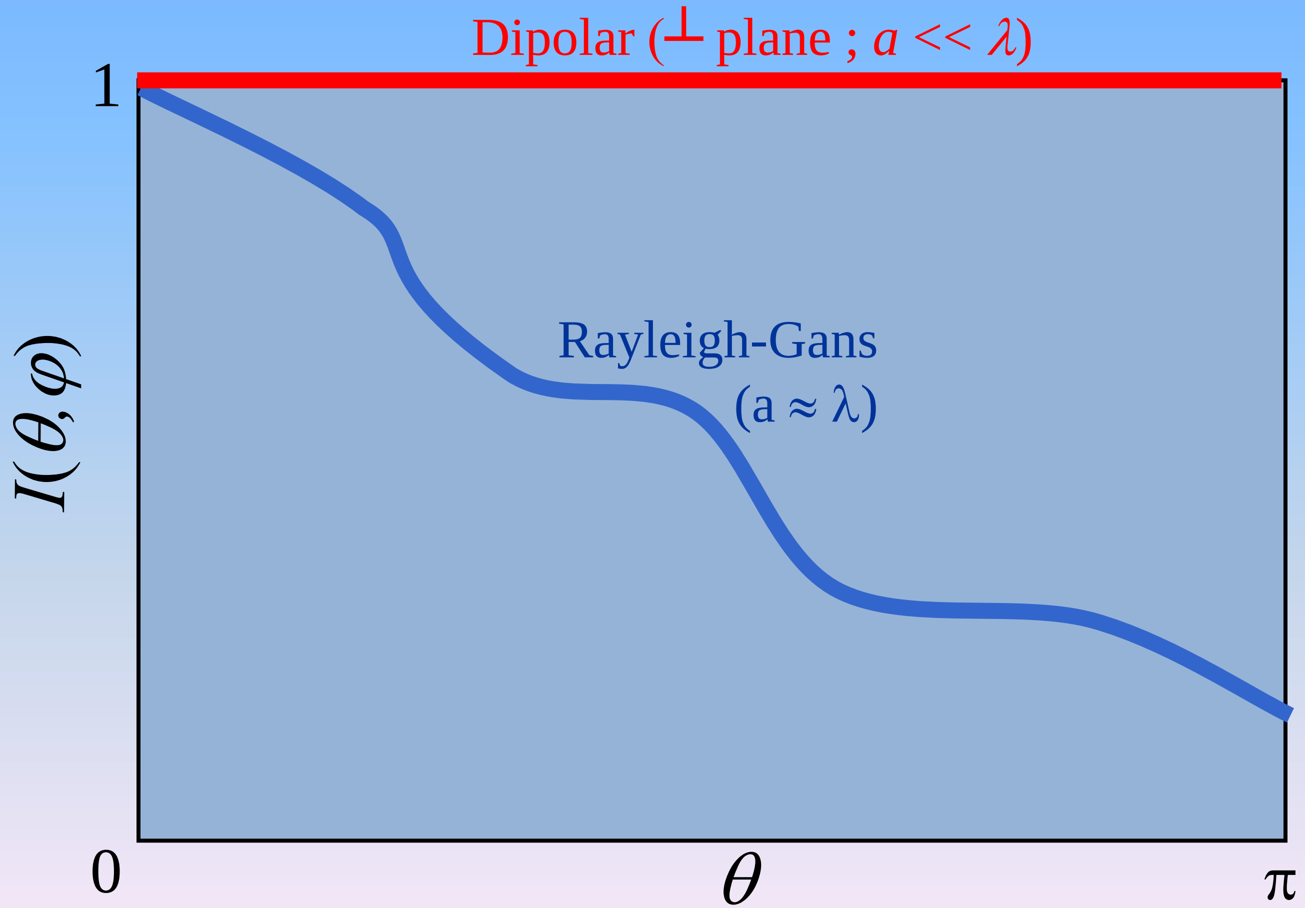
Rayleigh-Gans scattering (4)

EM D **RG** M
HG I PF
G B P F

In-plane Scattering Intensity $I(\theta,\varphi)$

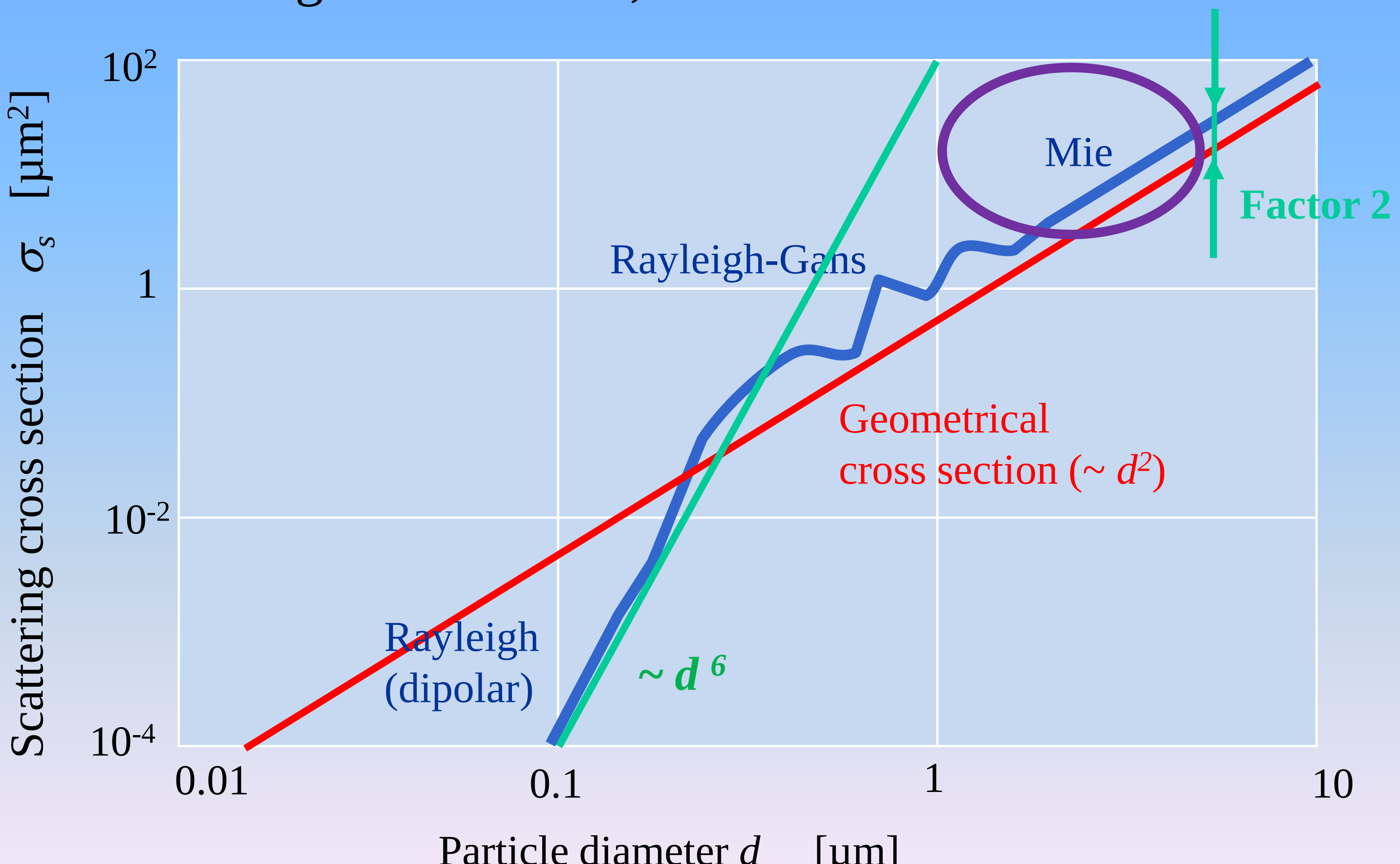
(vertical polarization)

a = particle radius



Optical properties: scattering

Scattering cross section, for λ = 500 nm and n = 1.5



Formalism of Mie-scattering (1)

EM
D
RG
M
HG
I
PF
G
B
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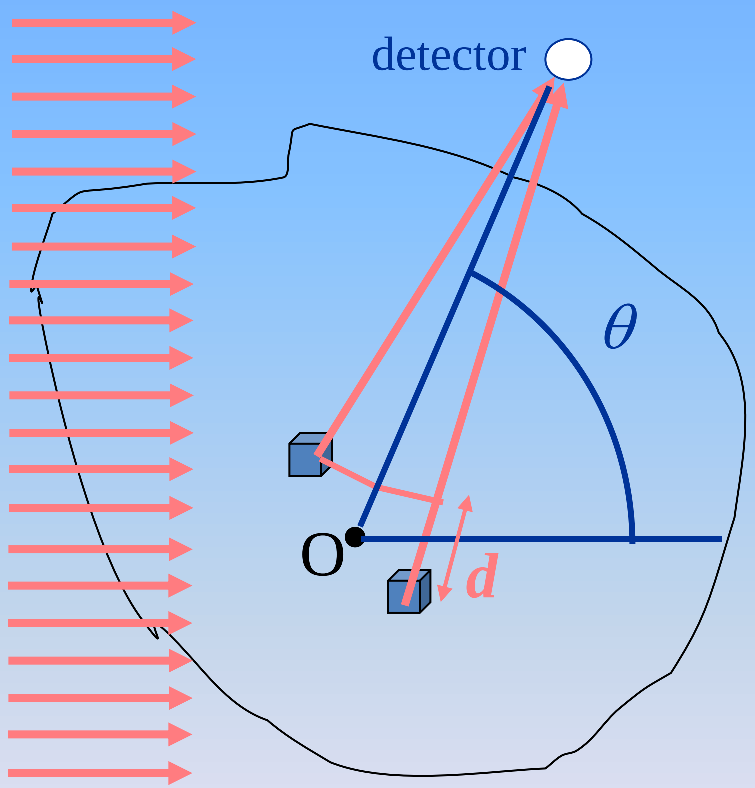
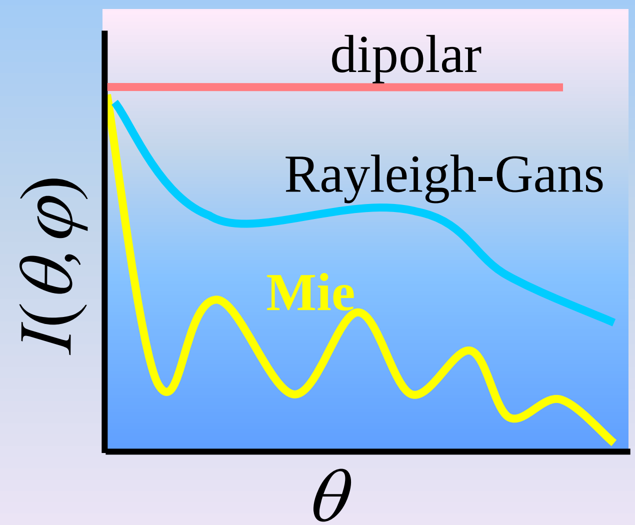
F

Large particles: $a >> \lambda$: ***(Rigorous) Mie-scattering***



Scattering from different parts will interfere at detector. Destructive interference will be dependent upon angle θ and upon particle shape.



Mie derived a rigorous treatment from Maxwell's equations; see below

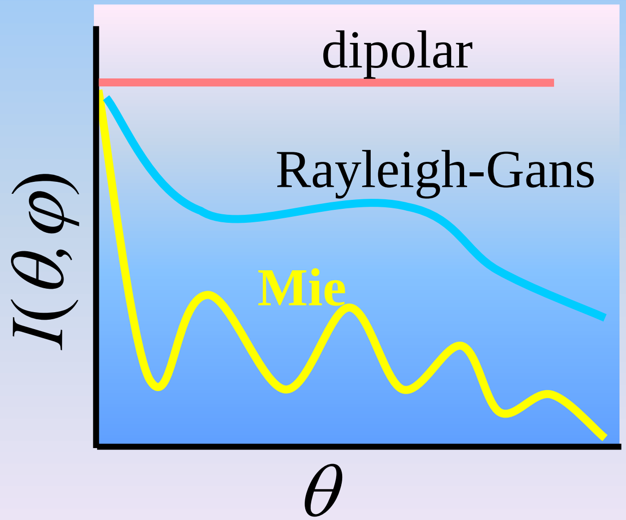
Formalism of Mie-scattering (2)

Large particles: $a >> \lambda$: ***(Rigorous) Mie-scattering***

Mie derived a rigorous treatment from Maxwell's equations;

For the derivation and resulting expressions for the field components $\boldsymbol{E}_\theta$ and $\boldsymbol{E}_\varphi$ see:

H.C. van de Hulst,
"Light scattering by small particles",
Sect. 9.2-3.



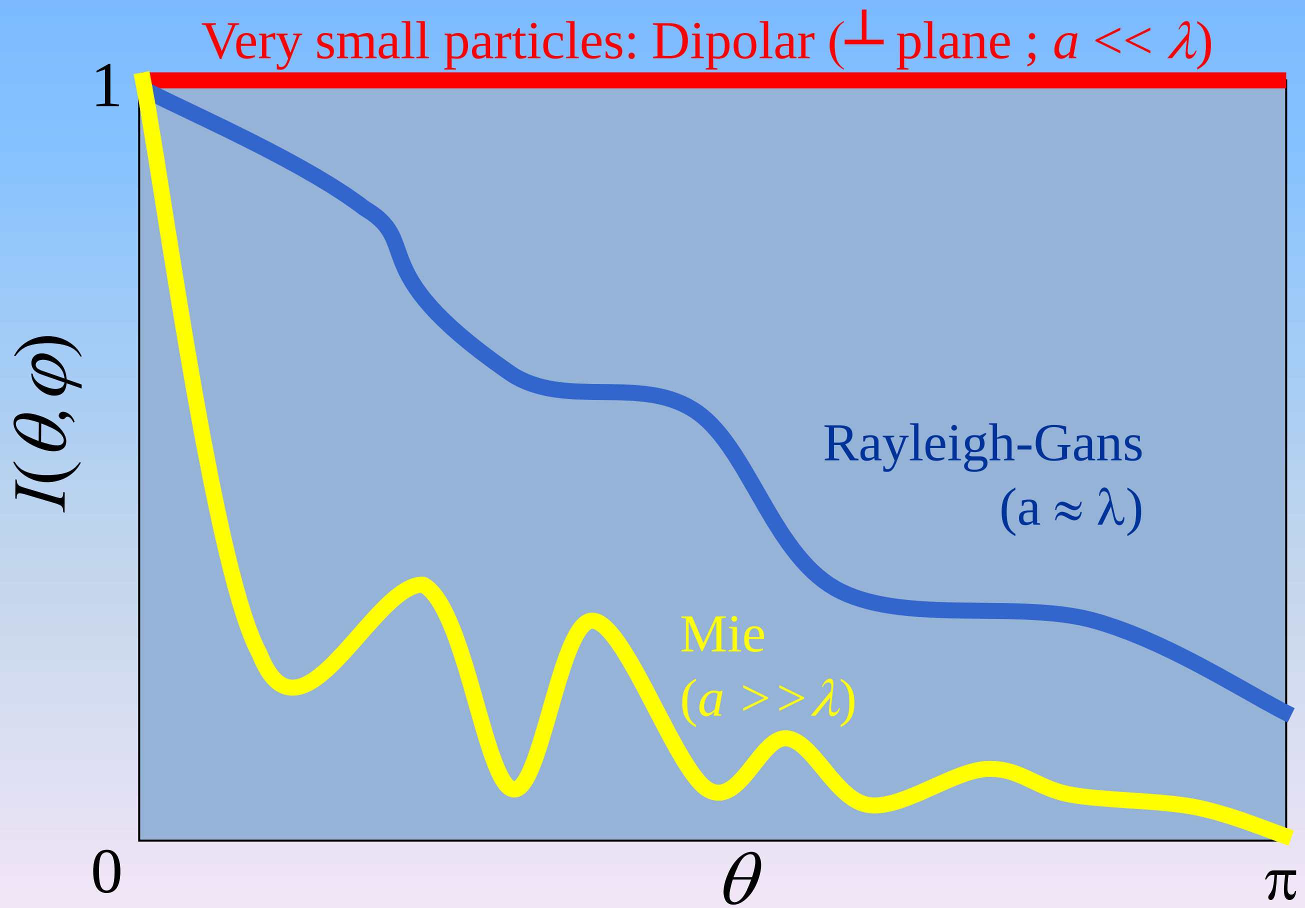
Formalism of Mie-scattering (3)

EM
D
RG
M
HG
I
PF
G
B
P
F

In-plane Scattering Intensity $I(\theta,\varphi)$

(vertical polarization)

a = particle radius



Other scattering functions (1)

EM
D
RG
M
HG
I
PF
G
B
P
F

- from astronomy:
  - Henyey-Greenstein: (radius a > ≈ 10 λ)

$$p(\theta,\varphi)=\frac{1}{4\pi}\frac{1-g^2}{\left[1+g^2-2g.\cos\theta\right]^{3/2}} \quad \text{with} \quad g=<\cos\theta>$$

  - Gegenbauer

- Isotropic:

$$p(\theta,\varphi)=\frac{1}{4\pi}$$

- Peaked-forward (*e.g*. Gaussian shape):

$$p(\theta,\varphi)=\frac{1}{4\pi}\exp(-\theta^2/\theta_0{}^2)$$

Other scattering functions (2)

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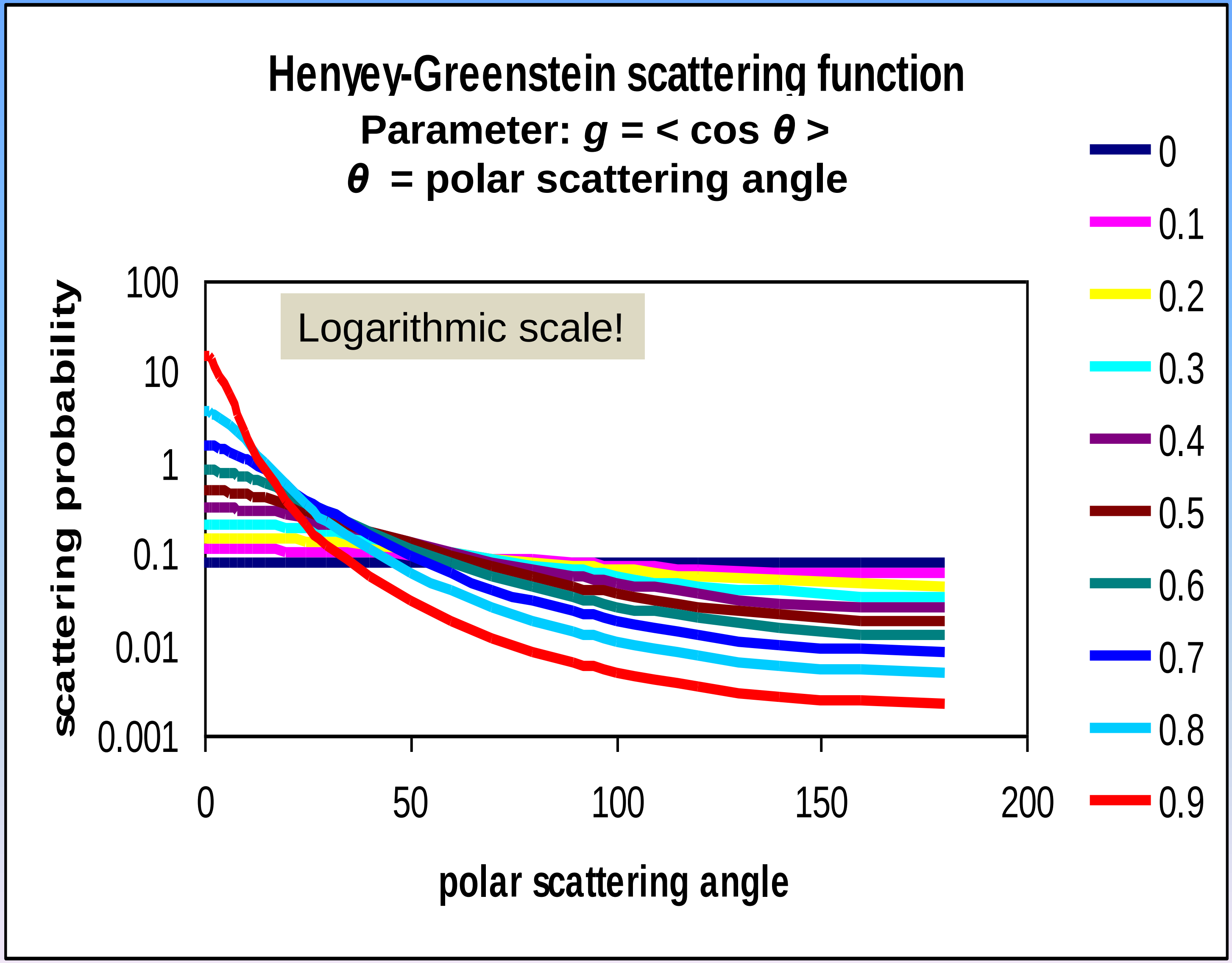
EM
D
RG
M

HG
I
PF

G
B
P
F



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Other scattering models

EM
D
RG
M

HG
I
PF

G
B
P
F

Other scattering models, all based on the Diffusion Equation (from Transport Equation):

G. Groenhuis, Ferwerda, Ten Bosch
- Scattering function: isotropic + peaked forward
- Source: pencil beam attenuating in tissue

B. Bonner, Nossal *et al.*
- 3D-grid of discrete scattering points
- Probabilistic approach

P. Patterson, Chance,Wilson
- Virtual point source for 1^{st} scatter event at depth $z_0 = 1/\mu_s'$.

F. Farrell, Patterson, Wilson
- Attenuating pencil beam, creating line of point sources.
- Each point source creates ”image source” above tissue surface (accounts for refr. index mismatch)

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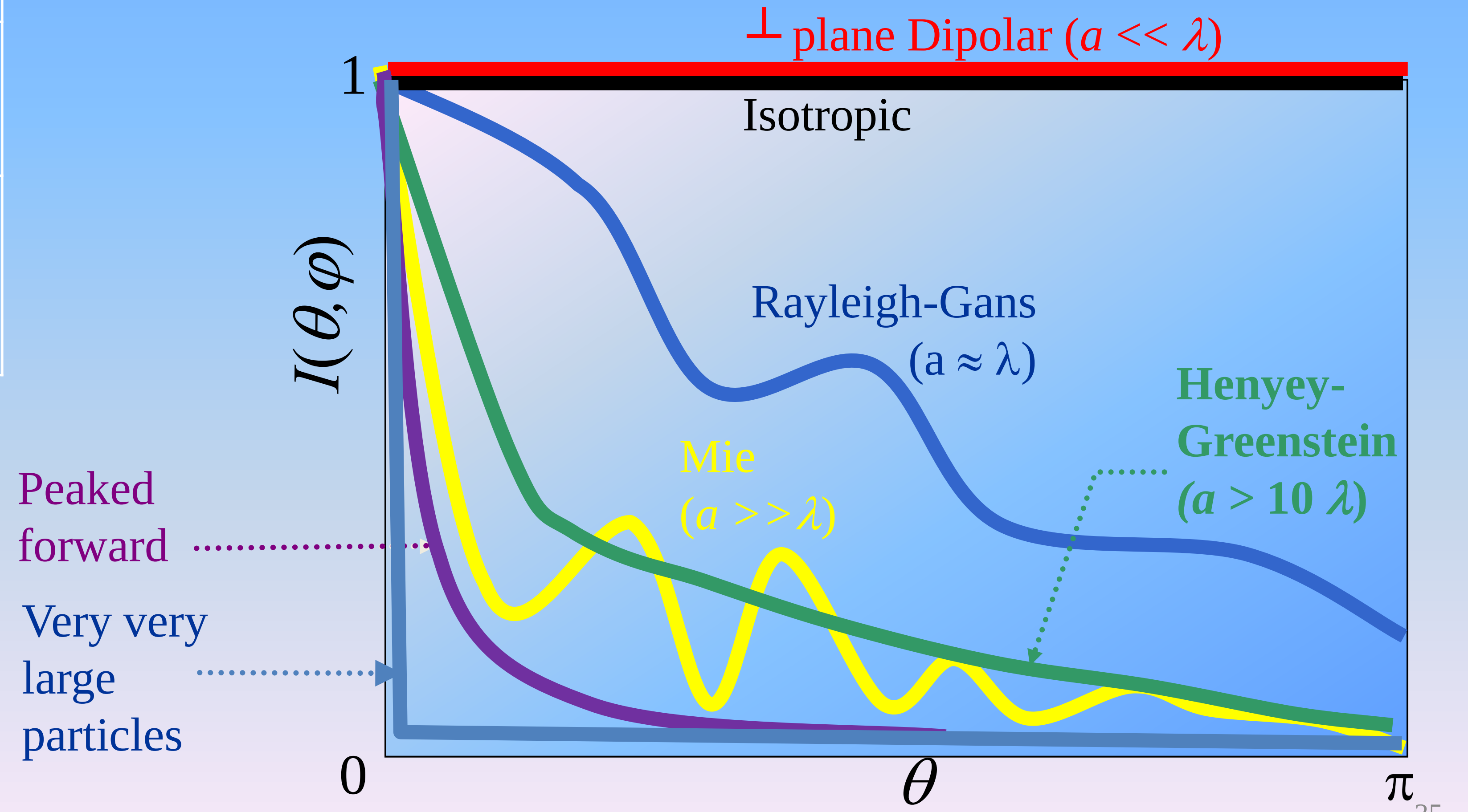
Overview of scattering functions

EM D RG M
HG I PF
G B P F

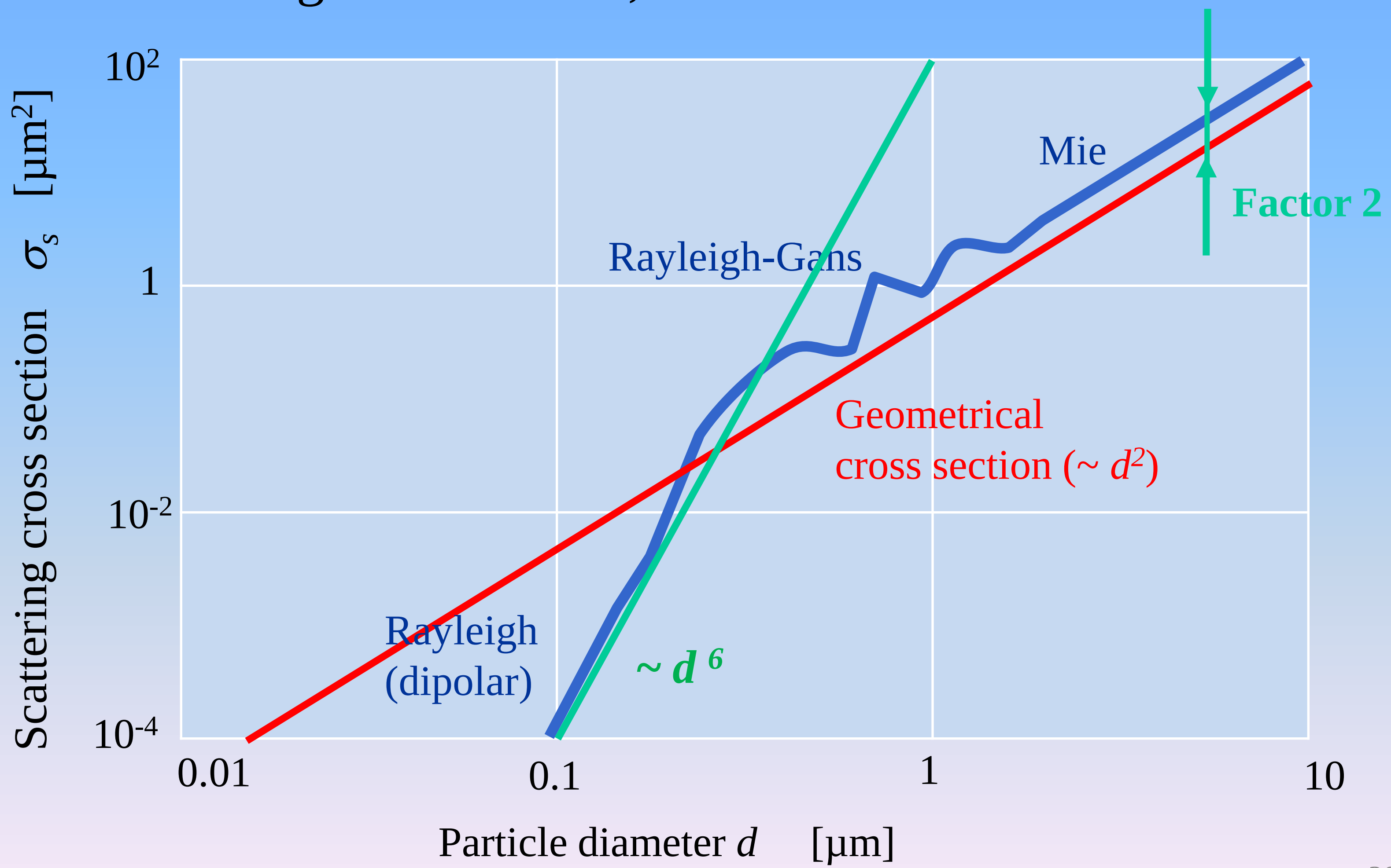
Scattering Intensity $I(\theta,\varphi)$

a = particle radius



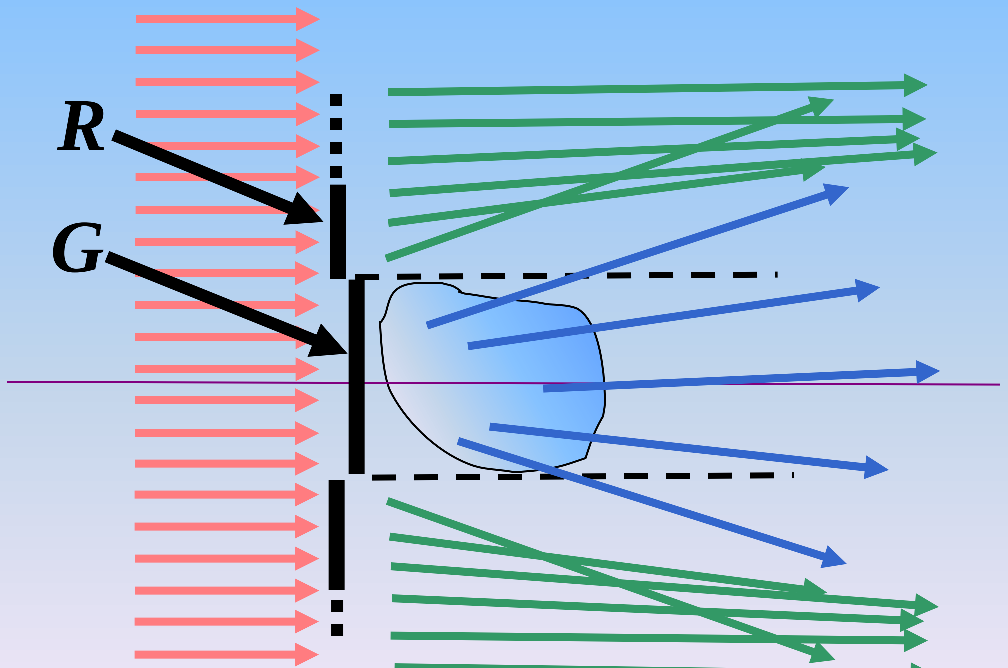
Optical properties: scattering

Scattering cross section, for λ = 500 nm and n = 1.5



Scattering by very large particles

Scattering by very large particles:
Scattering cross section = 2 x geometrical cross section.
Why ?



Scattering from inside particle will take away light due to geometrical cross section *G*

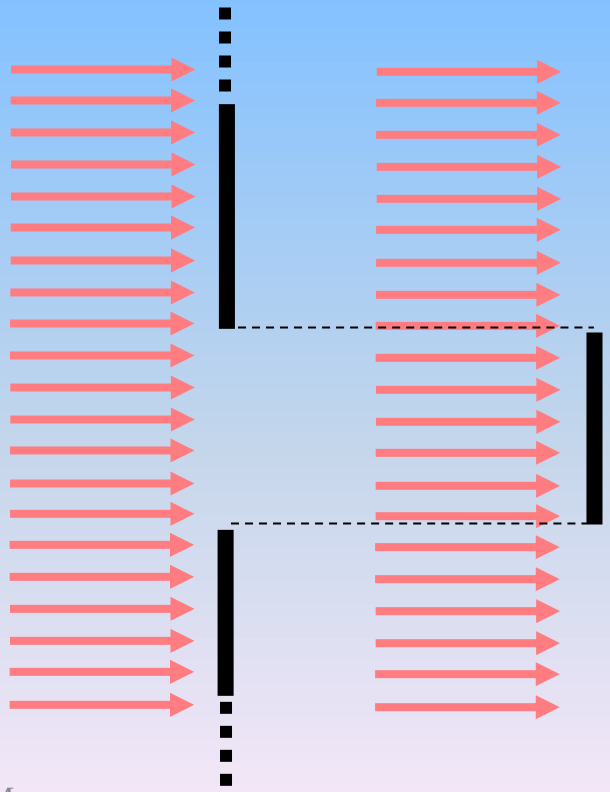
Diffraction around particle will take away light in a ring *R* equal to geometrical cross section *G*

"Extinction Paradox"

Very large particles: Extinction Paradox

Extinction Paradox:

Diffraction through hole = diffraction around obstruction, if hole and obstruction have identical shadow



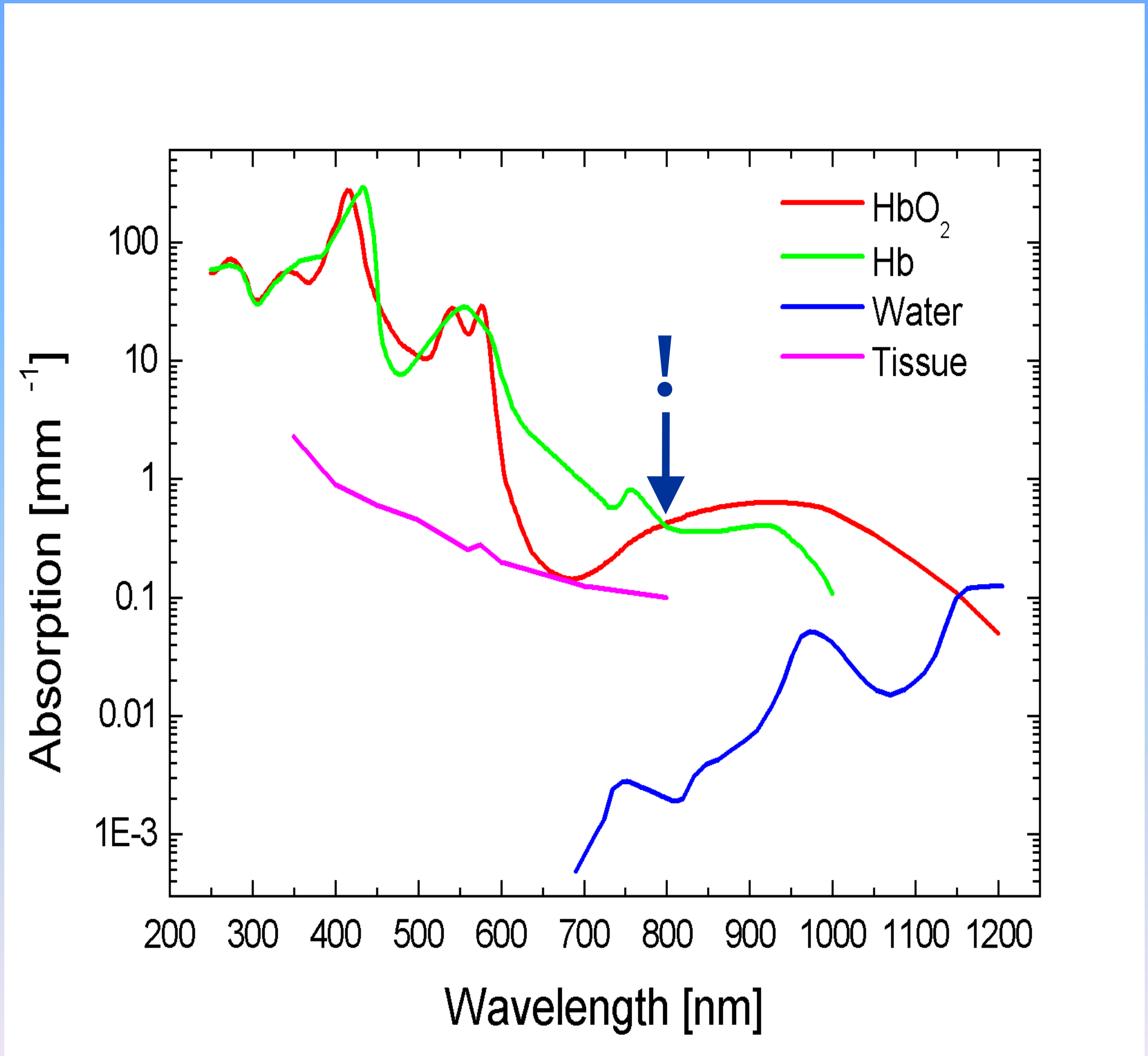
Original field + disturbance field behind:

- hole: E_{dist}
- obstruction: E_0 - E_{dist}

Intensity of disturbance field (= diffraction):

- hole: $I \sim E_{dist}^2$
- obstruction: $I \sim (-E_{dist})^2$

Optical properties of tissue and blood



(Reduced) **Scattering** coefficient: μ_s'

λ = 580 nm:

Dermis: 3 mm^{-1}

Blood: 1 …

- λ = 850 nm:

Dermis: 1 …

Blood: 0.5 …

! = Oxygen saturation measurements

based on differences in absorption for

λ > and < 800 nm

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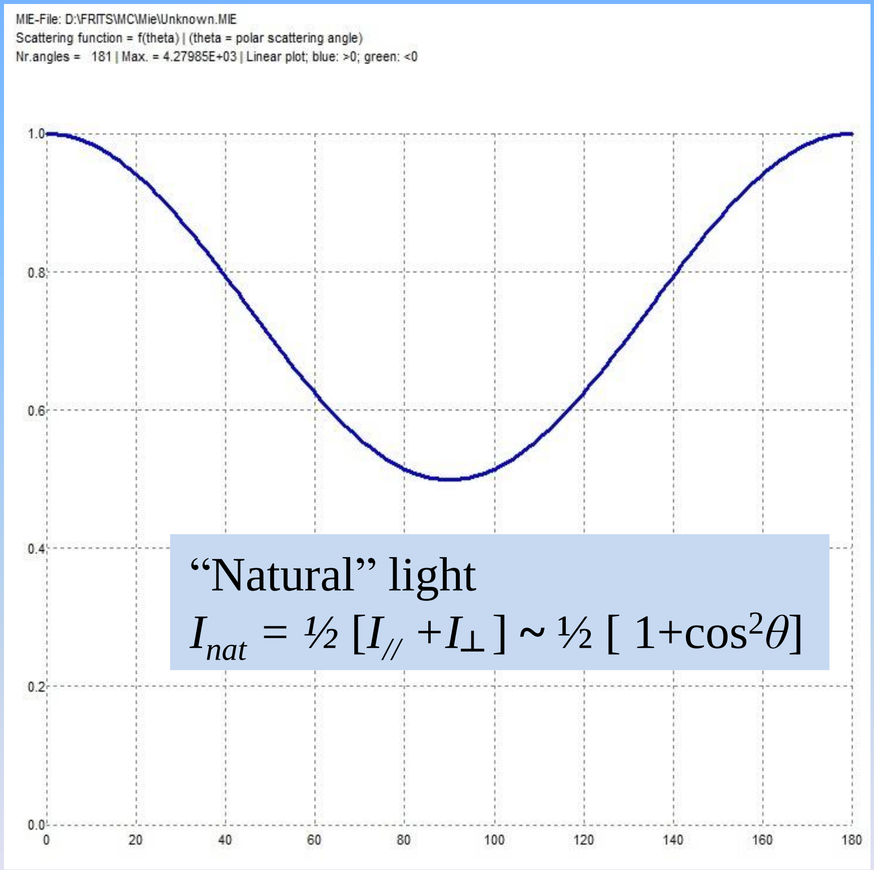
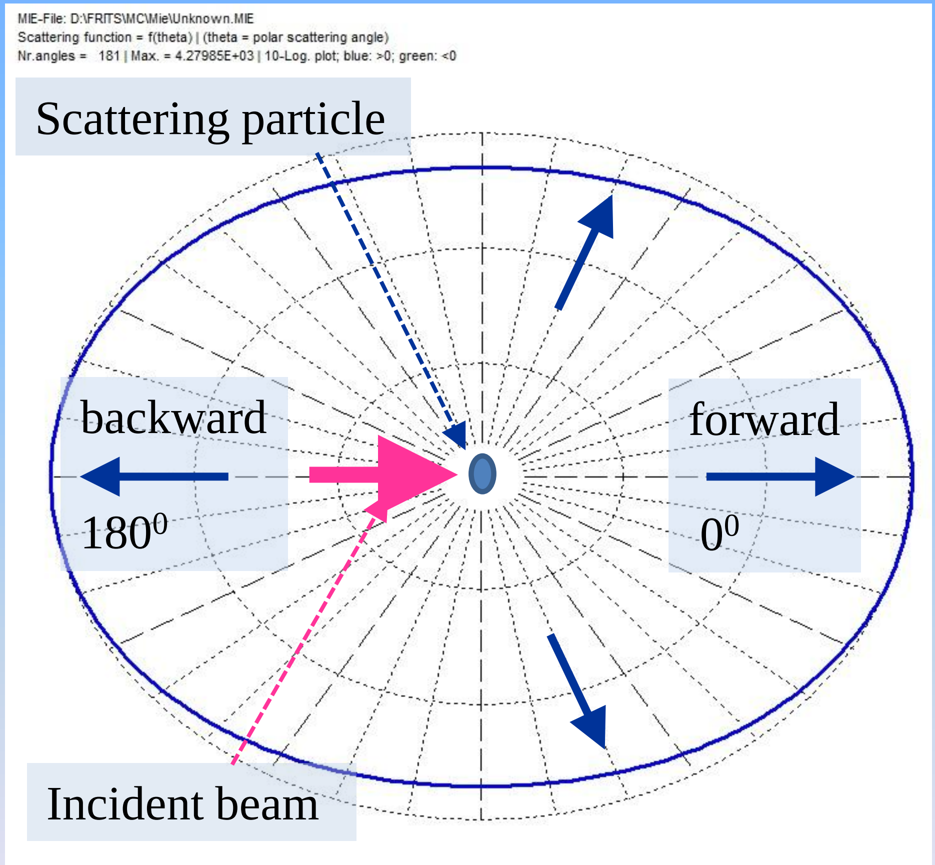
Examples of scattering functions

Dipolar scattering (in-plane and out-of-plane components summed)





Linear plot: linear scale:
X: 0 .. 180^0 ; Y: 0 .. 1

Polar plot: log. scale: 3 decades

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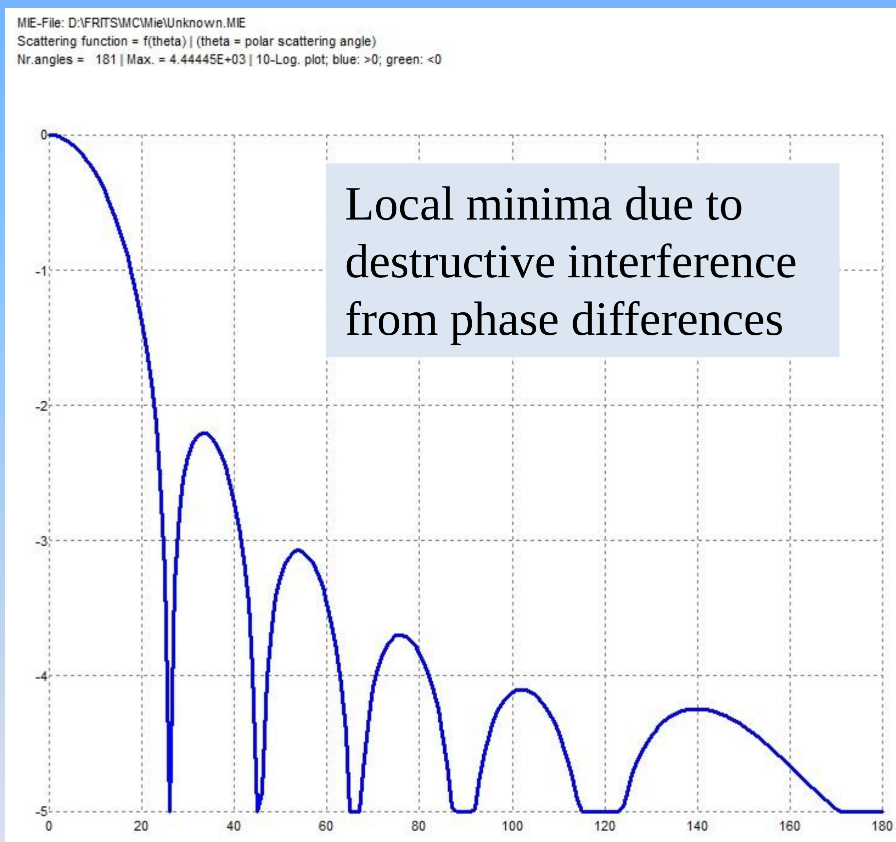
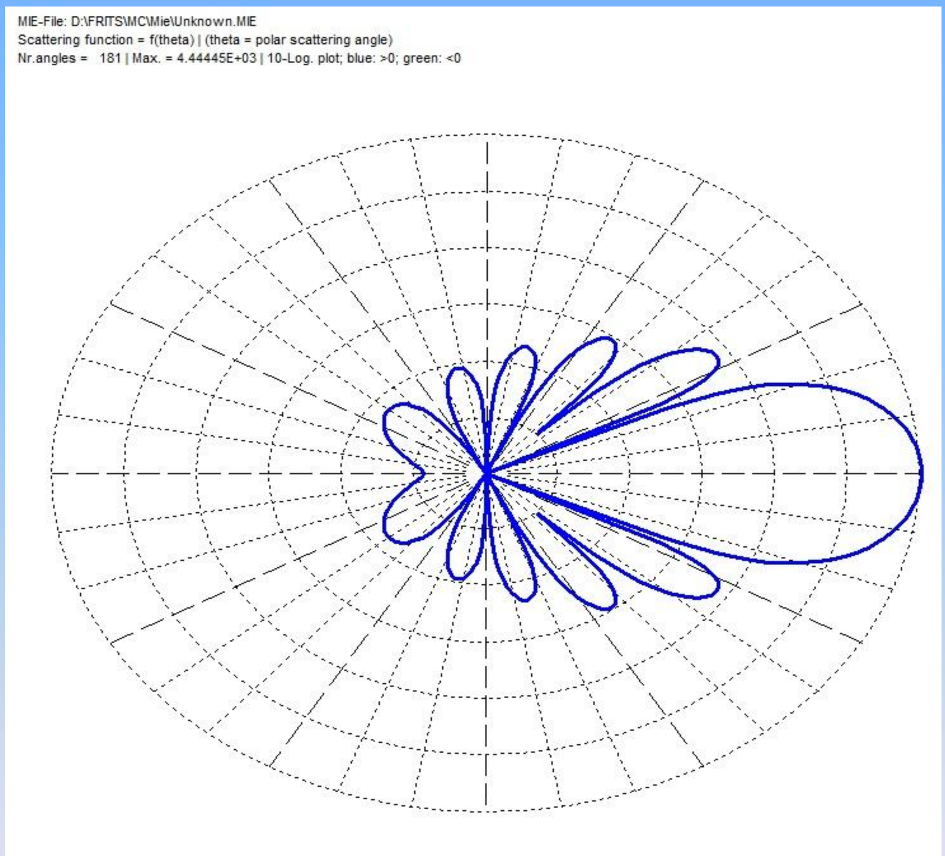
Examples of scattering functions

Rayleigh - Gans scattering



Local minima due to destructive interference from phase differences



Linear plot: log scale:
X: 0 .. 180^0 ; Y: 5 decades: 1 .. 10^{-5}

Polar plot: log. scale: 5 decades

Examples of scattering functions

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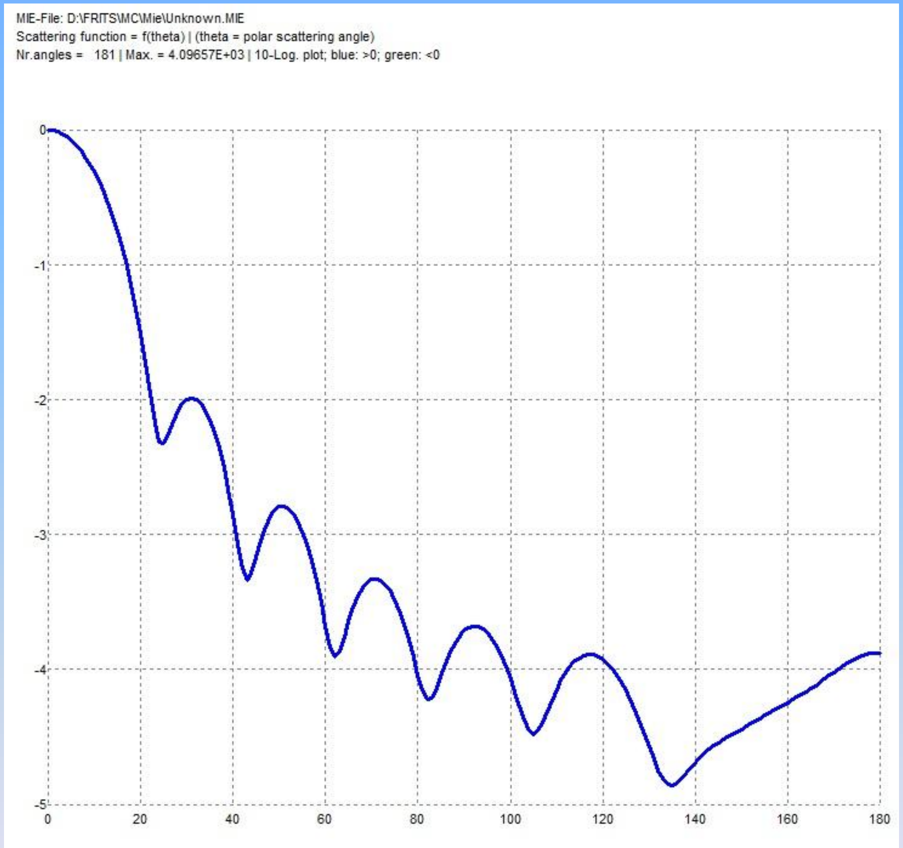
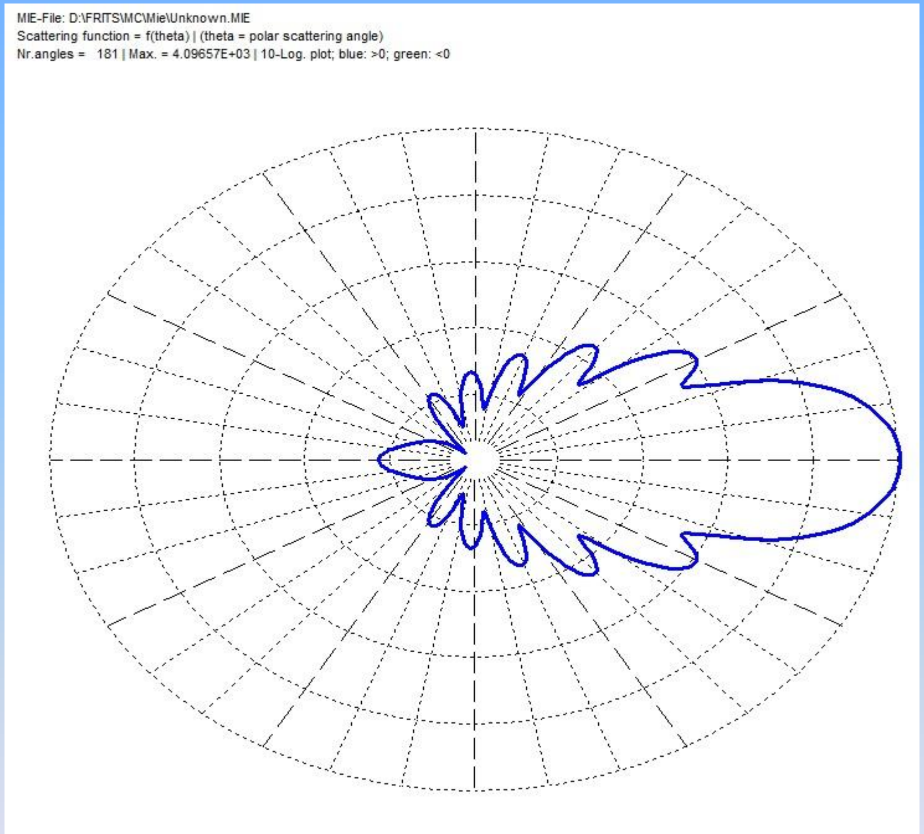
Mie scattering



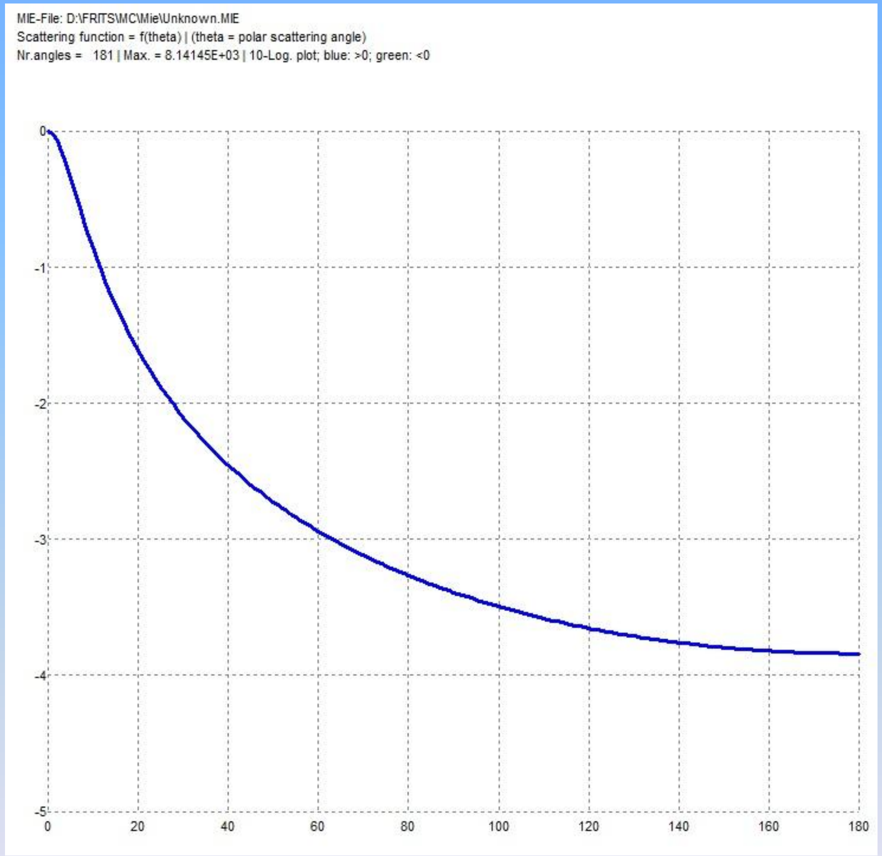


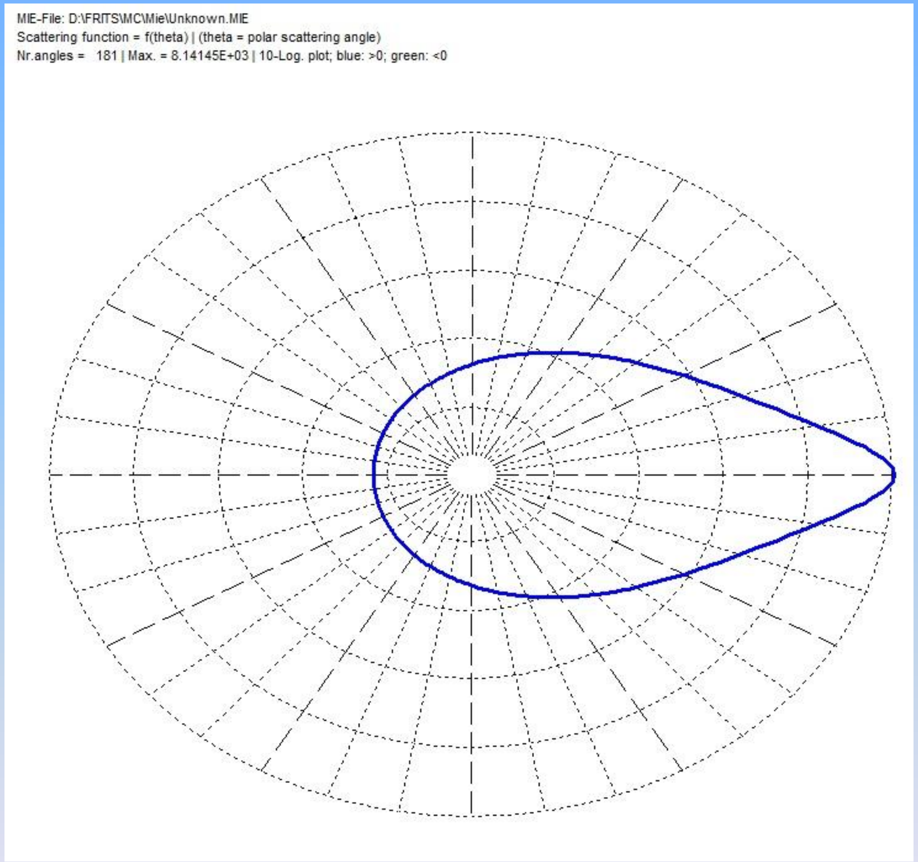
Linear plot: log scale:
X: 0 .. 180^0 ; Y: 5 decades: 1 .. 10^{-5}

Polar plot: log. scale: 5 decades

Examples of scattering functions

Henyey-Greenstein scattering





Linear plot: log scale:
X: 0 .. 180^0 ; Y: 5 decades: 1 .. 10^{-5}

Polar plot: log. scale: 5 decades

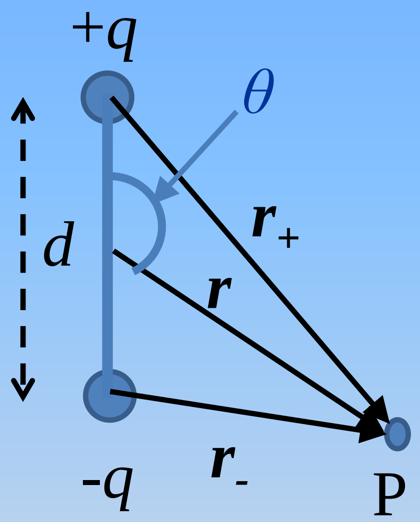
Appendices

Appendices:

A. Derivation of the dipole radiation formula

B. Derivation of the general scattering equation.

Appendix A: dipolar scattering



Dipole: dipole moment $p = qd$

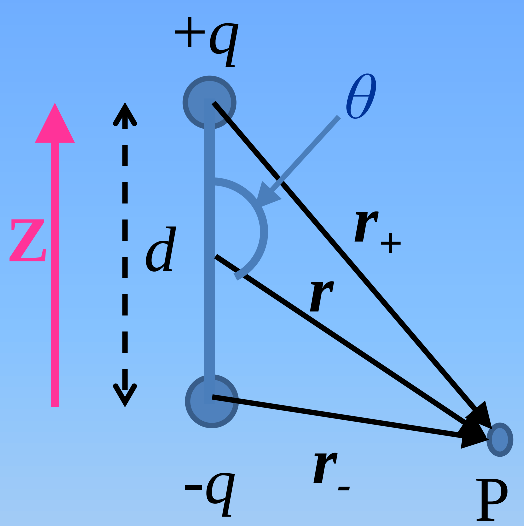
Oscillating dipole: $d(t) = d_0 .\cos \omega t$
$p(t) = p_0 .\cos \omega t$; $p_0 = qd_0$
(time-dependent part only)

We have to calculate the ***E***-field in point P:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

V = electric (scalar) potential
A = magnetic (vector) potential

Appendix A: dipolar scattering



z = coordinate along Z-axis;
assume $\boldsymbol{E_0}$ –field // Z-axis

F_e = electric force
m = mass
η = damping factor
ω_r = resonance frequency

Dipole: dipole moment $p = qd$

Oscillating dipole: $d(t) = d_0 . \cos \omega t$; $p_0 = qd_0$
(time-dependent part only)

Dipole, induced by $\boldsymbol{E}$-field:
classical harmonic oscillator :

$$\frac{d^2 z}{dt^2} + \eta \frac{dz}{dt} + \omega_r^{\,2} z = \frac{F_e}{m} = \frac{qE_0}{m} e^{i\omega t}$$

Solution:

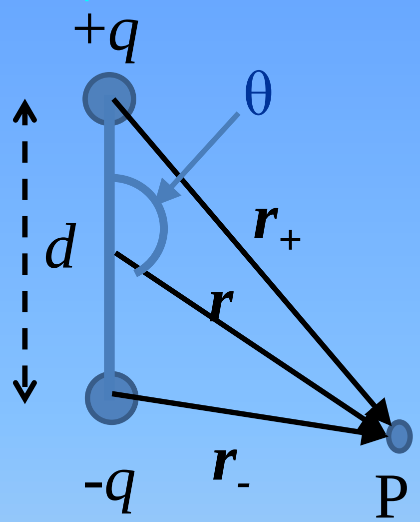
$$d(t) = 2z(t) = \frac{2qE_0}{m} \frac{e^{i\omega t}}{\omega_r^{\,2} - \omega^2 + i\eta\omega}$$

Electron free resonance frequency ω_r
>> induced frequency ω
(otherwise: if $\omega_r << \omega$: "Thomson scattering")

Induced dipole moment:

$$p(t) = p_0 \cos \omega t \; ; \; p_0 = \frac{2q^2 E_0}{m.\omega_r^{\,2}}$$

Appendix A: dipolar scattering



Dipole: dipole moment $p = qd$

Oscillating dipole: $p(t) = p_0 \cos \omega t$; $p_0 = q.d_0$

Now calculate the ***E***-field: $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$

We start with *V* ; later ***A***

Potential function of two charges (+ and **-**)

$$V(\vec{r},\theta,t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q\cos\{\omega(t - r_+/c)\}}{r_+} - \frac{q\cos\{\omega(t - r_-/c)\}}{r_-}\right]$$

with $q = \frac{p_0}{d_0}$

and $r_\pm = \sqrt{r^2 \mp rd.\cos\theta + (d/2)^2}$

Retarded time: ($t' = t - r_\pm /c$; c = light velocity) ,
accounts for flight time and phase differences between paths.

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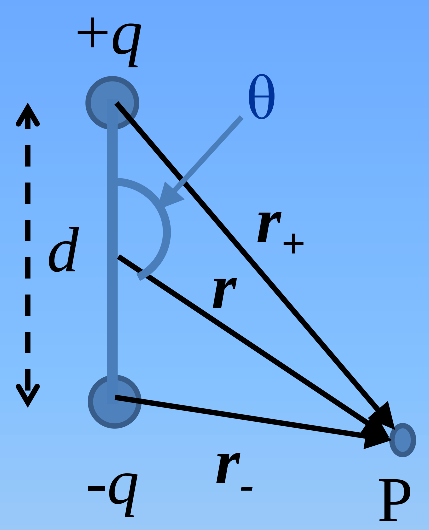
Appendix A: dipolar scattering



$$V(\vec{\boldsymbol{r}},\theta,t)=\frac{q}{4\pi\varepsilon_0}\left[\frac{\cos\{\omega(t-r_+/c)\}}{r_+}-\frac{\cos\{\omega(t-r_-/c)\}}{r_-}\right]$$

with $q=\dfrac{p_0}{d_0}$ and $r_\pm=\sqrt{r^2\mp rd.\cos\theta+(d/2)^2}$

Use **far-field approximation** ($d << r$; expansion to 1st order) :

$$r_\pm\approx r\left(1\mp\frac{d}{2r}\cos\theta+\left\{\frac{d}{2r}\right\}^2+\ldots\right)\Rightarrow\ \frac{1}{r_\pm}\approx\frac{1}{r}\left(1\pm\frac{d}{2r}\cos\theta\right)$$

Use **small-dipole approximation** ($d << \lambda = 2\pi.c/\omega \rightarrow \omega d/c<<1$; use $\cos(A\pm B)$):

$$\cos(A\pm B)=\cos A.\cos B\mp\sin A.\sin B\quad\rightarrow 1.\cos B\mp A.\sin B\ \text{ if }\ A\rightarrow 0$$

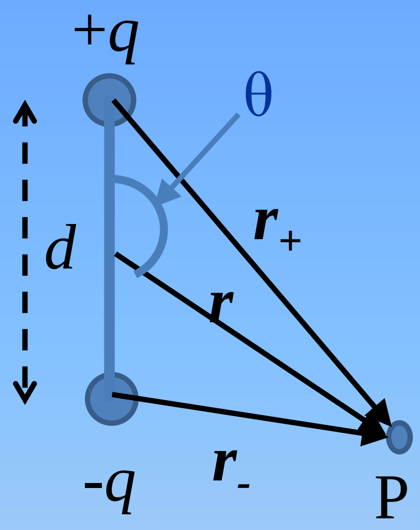
$$\cos[\omega(t-\frac{r_\pm}{c})]\approx\cos\left[\omega(t-\frac{r}{c})\pm\frac{\omega d}{2c}\cos\theta\right]\quad\approx 1.\cos[\omega(t-\frac{r}{c})]\mp\frac{\omega d}{2c}.\cos\theta\sin[\omega(t-\frac{r}{c})].$$

$$V(\vec{\boldsymbol{r}},\theta,t)=\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\{\omega(t-\frac{r}{c})\}+\frac{1}{r}\cos\{\omega(t-\frac{r}{c})\}\right]$$

Use **small-wavelength approximation** ($r >> \lambda = 2\pi.c/\omega$):

$$V(\vec{\boldsymbol{r}},\theta,t)=\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\{\omega(t-\frac{r}{c})\}\right]$$

Appendix A: dipolar scattering



Dipole: dipole moment $p = qd$

Oscillating dipole. $p(t) = p_0 \cos(\omega t)$; $p_0 = q_0 . d$

Retarded potential: (c = light velocity)

Calculation of V:

$$V(\vec{r},\theta,t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q\cos\{\omega(t - r_+/c)\}}{r_+} - \frac{q\cos\{\omega(t - r_-/c)\}}{r_-}\right]$$

with $r_\pm = \sqrt{r^2 \mp rd.\cos\theta + (d/2)^2}$

$$V(\vec{r},\theta,t) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\{\omega(t-\frac{r}{c})\} + \frac{1}{r}\cos\{\omega(t-\frac{r}{c})\}\right]$$

Result for potential V: ~ 1/***r*** :

$$V(\vec{r},\theta,t) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\{\omega(t-\frac{r}{c})\}\right]$$

Result for Electrostatic limit: ~ $1/r^2$:

$$\omega \to 0 : V(\vec{r},\theta,t) \to \frac{p_0\cos\theta}{4\pi\varepsilon_0 r^2}$$

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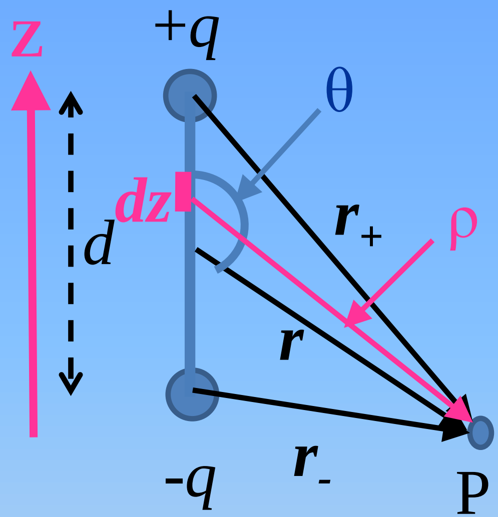
Appendix A: dipolar scattering

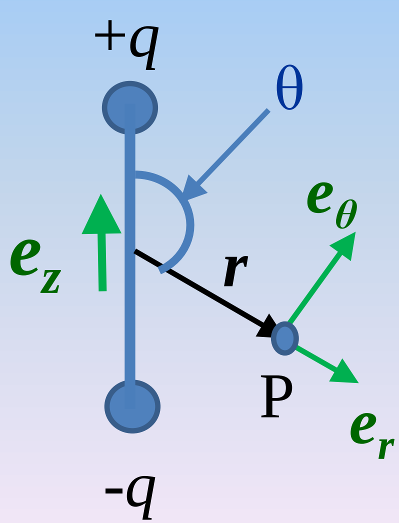


***E*-field:** $\vec{E} = -\nabla V - \dfrac{\partial \vec{A}}{\partial t}$

Now calculate the ***A*-field (vector potential):**

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{r}\,dl = \frac{\mu_0}{4\pi}\int_{-d_0/2}^{d_0/2} \frac{-q\omega \sin[\omega(t-\rho/c)\}}{r}\,\vec{e}_z dz$$

$$\text{with } \vec{I} = \frac{dq}{dt}\vec{e}_z = -q\omega \sin(\omega t)\,\boldsymbol{e}_z \quad ; \quad q = \frac{p_0}{d_0} \quad \text{and} \quad \rho = f(z)$$



We want the result at large distances,
so we use as an approximation:
replace the integrand by its value at center O,
multiply with 2. $d_0/2 = d_0$ and take $d_0\, q = p_0$.

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\{\omega(t-\frac{r}{c})\}\,\boldsymbol{e}_z$$

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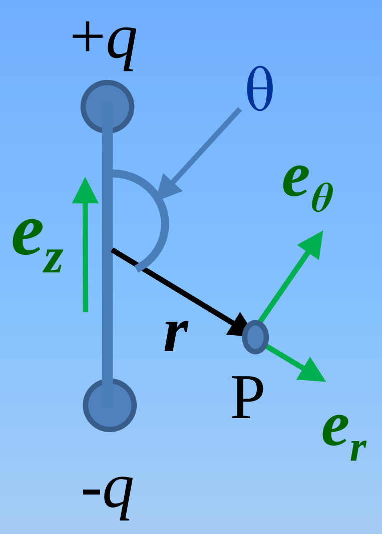
Appendix A: dipolar scattering



$$V(\vec{r},\theta,t) = -\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\,\frac{\omega}{c}\sin\{\omega(t-\frac{r}{c})\}$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\{\omega(t-\frac{r}{c})\}\boldsymbol{e}_z$$

Now calculate the ***E***-field:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

with $\nabla V = \frac{\partial V}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\vec{e}_\theta$

and $\vec{e}_z = \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta$

Final result:

$$\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\,\frac{\sin\theta}{r}\cos\left[\omega\{t-\frac{r}{c}\}\right]\vec{e}_\theta$$

E-field: $\sim E_0$ (incident field)
$\sim \omega^2$ ($\rightarrow$ intensity $\sim \omega^4$)
$\sim r^{-1}$ ($\rightarrow$ intensity $\sim r^{-2}$)
θ – component only !
retarded time.

NB. *V*-field renders ***r***-component only!
A-field renders both ***r*** and ***θ***-components;
and in resulting ***E***-field both ***r***-components cancel ;
θ-component remains only!

Appendix A: dipolar scattering

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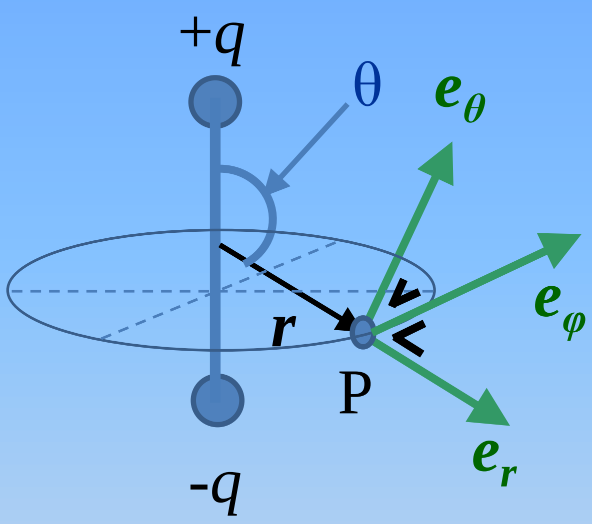


What is emitted energy?

Radiated power [W/m²] :
Poynting vector $\boldsymbol{S}$:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\frac{\sin\theta}{r}\cos\left[\omega\{t-\frac{r}{c}\}\right]\vec{e}_\theta$$

$$p_0 = \frac{2q^2 E_0}{m.\omega_r^2}$$

Analogously:

$$\vec{B} = \nabla\times\vec{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi\, c}\frac{\sin\theta}{r}\cos\left[\omega\{t-\frac{r}{c}\}\right]\vec{e}_\varphi$$

with $\vec{A}(\vec{r},t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\{\omega(t-\frac{r}{c})\}\boldsymbol{e}_z$

If $\vec{X} \,//\, \vec{e}_z$: $\nabla\times\vec{X} =$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(rX_\theta) - \frac{\partial X_r}{\partial\theta}\right]\vec{e}_\varphi$$

using $\vec{e}_z = \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta$

$\boldsymbol{B}$ // $\boldsymbol{e}_\varphi$: in ***tangential*** direction → Ampère's Law.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\vec{e}_r$$

NB. ***S*** in ***radial*** direction!

Appendix A: dipolar scattering

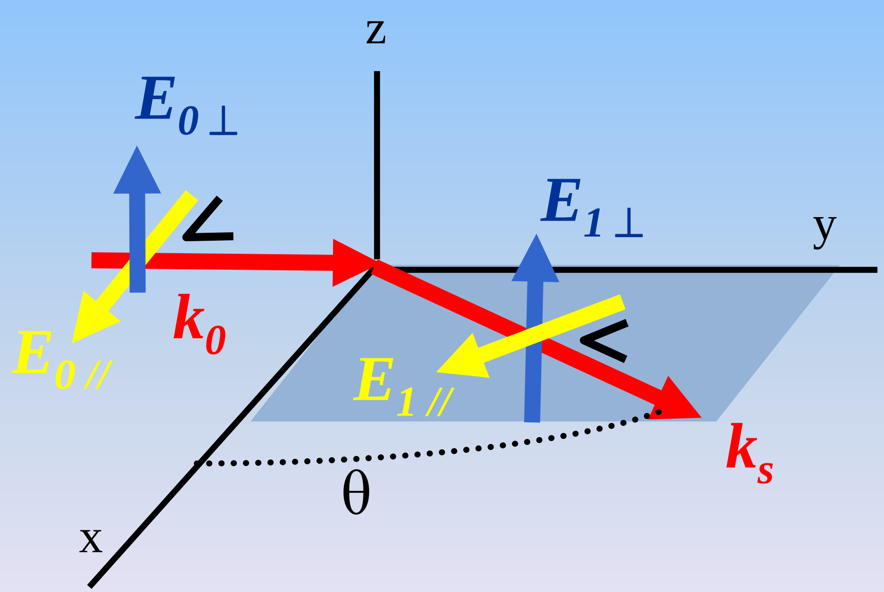
Final result for 1 dipole:

$$\vec{\mathbf{E}} = \frac{\mu_0 p_0 \omega^2}{4\pi} \frac{\sin\theta}{r} \cos\left[\omega\{t - \frac{r}{c}\}\right] \vec{\mathbf{e}}_\theta$$

$$p_0 = \frac{2q^2 E_0}{m.\omega_r{}^2}$$

$$\text{with } k = \frac{\omega}{c} \text{ and } \varepsilon_0 \mu_0 = c^{-2} \ : \ \vec{\mathbf{E}} = \frac{p_0 k^2}{4\pi\varepsilon_0} \frac{\sin\theta}{r} \cos\left[\omega\{t - \frac{r}{c}\}\right] \vec{\mathbf{e}}_\theta$$

For ⊥ component: $\sin\theta = 1$



Expression for dipoles in part I: (amplitude)

$$\begin{bmatrix} \mathbf{E}_{1\perp} \\ \mathbf{E}_{1//} \end{bmatrix} \propto \frac{\alpha\, k_s{}^2}{4\pi\varepsilon_0 r} \begin{bmatrix} 1 & \\ & \sin\theta \end{bmatrix} \begin{bmatrix} \mathbf{E}_{0\perp} \\ \mathbf{E}_{0//} \end{bmatrix}$$

Correspondence if set: $p_0 = \alpha\, E_0$

Appendices

Appendices:

A. Derivation of the dipole radiation formula

B. Derivation of the general scattering equation

Appendix B: scattering field

Electromagnetic Theory from Maxwell's Equations $\rho = 0 \; ; \; j = 0$

$$\nabla \bullet \vec{\boldsymbol{D}} = 0 \quad ; \quad \nabla \bullet \vec{\boldsymbol{B}} = 0 \quad ; \quad \nabla \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t} \quad ; \quad \nabla \times \vec{\boldsymbol{B}} = \mu_0 \frac{\partial \vec{\boldsymbol{D}}}{\partial t}$$

Wave Equation: $$\nabla \times \nabla \times \vec{\boldsymbol{E}} = -\mu_0 \frac{\partial^2 \vec{\boldsymbol{D}}}{\partial \boldsymbol{t}^2}$$

$\nabla \times \vec{\boldsymbol{E}} =$ "rotation" , "curl"

Assume: scattering field is small compared with incident field

$$\vec{\boldsymbol{D}} = \varepsilon . \vec{\boldsymbol{E}}, \quad \text{with} \quad \vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_0 + \vec{\boldsymbol{E}}_1$$

$$\varepsilon = \varepsilon_0 + \varepsilon_1 \; ; \qquad \varepsilon_1 = \varepsilon_0 (\varepsilon_r - 1)$$

All $\boldsymbol{D}$'s , $\boldsymbol{E}$'s and $\boldsymbol{\varepsilon}$'s are $f\,(\boldsymbol{r},t)$

Assume: $\varepsilon_1 << \varepsilon_0$ and $\boldsymbol{E_1} << \boldsymbol{E_0}$; **1^{st} order approximation**:
0^{th} and 1^{st} order terms only:

$$\vec{\boldsymbol{D}}_0 = \varepsilon_0 . \vec{\boldsymbol{E}}_0$$

$$\vec{\boldsymbol{D}}_1 = \varepsilon_0 . \vec{\boldsymbol{E}}_1 + \varepsilon_1 . \vec{\boldsymbol{E}}_0$$

Appendix B: scattering field

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Wave Equation:

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$

All $\boldsymbol{D}$'s , $\boldsymbol{E}$'s and $\boldsymbol{\varepsilon}$'s are $f(\boldsymbol{r},t)$

$$\vec{D}_0 = \varepsilon_0 . \vec{E}_0$$

$$\vec{D}_1 = \varepsilon_0 . \vec{E}_1 + \varepsilon_1 . \vec{E}_0$$

$$\varepsilon_1 = (\varepsilon_r - 1)\varepsilon_0$$

$$\text{0th order}: \quad \nabla \times \nabla \times \vec{E}_0 = -\nabla^2 \vec{E}_0 + \nabla(\nabla \bullet \vec{E}_0) = -\varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}_0}{\partial t^2}$$

$$\nabla \bullet \vec{E}_0 = 0: \quad \nabla^2 \vec{E}_0 = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}_0}{\partial t^2}$$

Solution: incident field from light source:

$$\vec{E}_0(\vec{r},t) = \vec{E}_{0m} \exp\left[i(\vec{k}_0 \bullet \vec{r} - \omega_0 t)\right]$$

harmonic wave:

$k = 2\pi/\lambda$; $\omega = 2\pi f$

$$\text{1st order}: \quad \nabla^2 \vec{D}_1 - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{D}_1}{\partial t^2} = -\nabla \times \nabla \times (\varepsilon_1 \vec{E}_0)$$

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Appendix B: scattering field

$$\text{1st order :}\quad \nabla^2 \vec{\boldsymbol{D}}_1 - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{\boldsymbol{D}}_1}{\partial t^2} = -\nabla\times\nabla\times(\varepsilon_1 \vec{\boldsymbol{E}}_0)$$

ε_1 , $\boldsymbol{E}_0$ and $\boldsymbol{E}_1$ are $f(\boldsymbol{r},t)$;
ε_0 = constant

Derivation 1st order:

$$\nabla^2 \vec{\boldsymbol{D}}_1 = \nabla^2(\varepsilon_0 \vec{\boldsymbol{E}}_1 + \varepsilon_1 \vec{\boldsymbol{E}}_0) =$$

$$= -\nabla\times\nabla\times(\varepsilon_0 \vec{\boldsymbol{E}}_1) - \nabla\times\nabla\times(\varepsilon_1 \vec{\boldsymbol{E}}_0) + \nabla(\nabla\bullet(\varepsilon_0 \vec{\boldsymbol{E}}_1)) + \nabla(\nabla\bullet(\varepsilon_1 \vec{\boldsymbol{E}}_0)) =$$

$$\text{use :}\, \nabla\bullet\vec{\boldsymbol{D}}_1 = \nabla\bullet(\varepsilon_0 \vec{\boldsymbol{E}}_1 + \varepsilon_1 \vec{\boldsymbol{E}}_0) = 0$$

$$= -\varepsilon_0 \nabla\times\nabla\times\vec{\boldsymbol{E}}_1 - \nabla\times\nabla\times(\varepsilon_1 \vec{\boldsymbol{E}}_0) =$$

$$= +\varepsilon_0 \mu_0 \tfrac{\partial^2}{\partial t^2}(\varepsilon_0 \vec{\boldsymbol{E}}_1) - \nabla\times\nabla\times(\varepsilon_1 \vec{\boldsymbol{E}}_0) = \qquad (\text{since } \nabla\bullet\vec{\boldsymbol{E}}_1 = 0)$$

And so:

$$\nabla^2 \vec{\boldsymbol{D}}_1 = \varepsilon_0 \mu_0 \tfrac{\partial^2}{\partial t^2} \vec{\boldsymbol{D}}_1 - \nabla\times\nabla\times(\varepsilon_1 \vec{\boldsymbol{E}}_0)$$

At the detector:
no incident field present!
There:

$\boldsymbol{D}_1 = \varepsilon_0 \boldsymbol{E}_1 + \varepsilon_1 \boldsymbol{E}_0 \rightarrow \varepsilon_0 \boldsymbol{E}_1$

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Appendix B: scattering field

Wave Equation:

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$\vec{D}_0 = \varepsilon_0 . \vec{E}_0 \quad ; \quad \varepsilon_0, \varepsilon_1, \vec{D} \text{ and } \vec{E} \text{ are } f(\vec{r}, t)$$

$$\vec{D}_1 = \varepsilon_0 \vec{E}_1 + \varepsilon_1 \vec{E}_0$$

$$\text{1st order :} \quad \nabla^2 \vec{D}_1 - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{D}_1}{\partial t^2} = -\nabla \times \nabla \times (\varepsilon_1 \vec{E}_0)$$

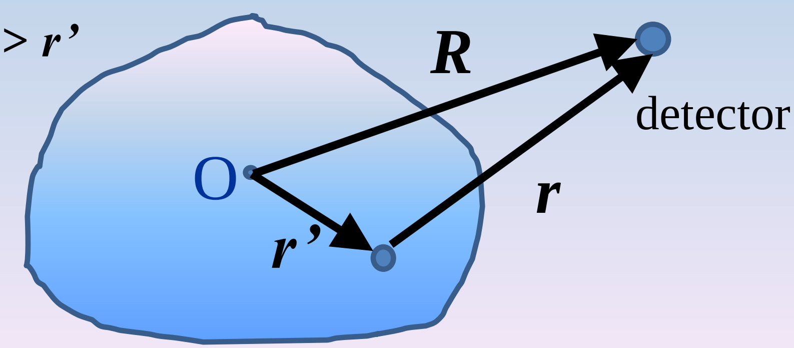
$$\text{Define } \vec{Z} \text{ using } : \vec{D}_1 = \nabla \times \nabla \times \vec{Z}$$

$$\nabla^2 \vec{Z} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{Z}}{\partial t^2} = -\varepsilon_1 \vec{E}_0$$

Solution using Green's functions:

$$\vec{Z}(\vec{R}, t) = \frac{1}{4\pi} \iiint d^3\vec{r}\,' \frac{\varepsilon_1 \vec{E}_0(\vec{r}\,', t')}{\left|\vec{R} - \vec{r}\,'\right|}$$

Scattering volume: $R >> r'$



t' = retarded time: $t' = t - |\mathbf{R}-\mathbf{r}|/c$

c = light velocity: $c = 1 / \sqrt{(\varepsilon_0 \mu_0)}$

t' accounts for flight time to detector

(Solution explained later)

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Appendix B: scattering field

$\vec{Z}$ is defined using :

$$\vec{D}_1 = \nabla \times \nabla \times \vec{Z}$$

Solution using Green's functions:

$$\vec{Z}(\vec{R},t) = \frac{1}{4\pi} \iiint d^3\vec{r}\,' \frac{\varepsilon_1 \vec{E}_0(\vec{r}\,',t')}{\left|\vec{R}-\vec{r}\,'\right|}$$

with : $\vec{E}_0(\vec{r}\,',t') = \vec{E}_{0m} \exp\left[i(\vec{k}_{\vec{r}} \bullet \vec{r}\,' - \omega_0 t')\right]$

$$\vec{D}_1 = \frac{1}{4\pi} \iiint d^3\vec{r}\,' . \nabla \times \nabla \times \frac{\varepsilon_1 \vec{E}_0(\vec{r}\,',t')}{\left|\vec{R}-\vec{r}\,'\right|}$$

At detector no incoming field present:

$\boldsymbol{D_1} = \varepsilon_0 \boldsymbol{E_1} + \varepsilon_1 \boldsymbol{E_0} \;\rightarrow\; \varepsilon_0 \boldsymbol{E_1}$

using : $\iiint d^3\vec{r}\,' . \nabla \times \nabla \times f(e^{i\vec{k}\bullet\vec{r}\,'}, \vec{r}\,') = \iiint d^3\vec{r}\,' . \vec{k} \times \vec{k} \times f(e^{i\vec{k}\bullet\vec{r}\,'}, \vec{r}\,')$

f denotes any function of the arguments

(after volume integration, $\boldsymbol{k}$ is the only coordinate-dependent variable;
$\boldsymbol{k} = f(x,y,z)$; $\boldsymbol{r}' = f(x',y',z'$; integration and differentiation can be swapped)

$$\varepsilon_0 \vec{E}_1 = \frac{1}{4\pi} \iiint d^3\vec{r}\,' . \vec{k} \times \vec{k} \times \frac{\varepsilon_1 \vec{E}_0(\vec{r}\,',t')}{\left|\vec{R}-\vec{r}\,'\right|}$$

- retarded time : accounts for flight time from scattering volume to detector
- ($\boldsymbol{k}$ x $\boldsymbol{k}$ x ..) operator : accounts for polarization directions of $\boldsymbol{E_1}$ from $\boldsymbol{E_0}$.
- $|\boldsymbol{R}-\boldsymbol{r}'|$ in denominator: spherical wave; intensity (~ $|\boldsymbol{E_1}|^2$) is ~ $1/R^2$.

Appendix B: scattering field

Solution using Green's functions:

$$\vec{Z}(\vec{R},t)=\frac{1}{4\pi}\iiint d^3\vec{r}'\frac{\varepsilon_1\vec{E}_0(\vec{r}',t')}{\left|\vec{R}-\vec{r}'\right|}$$

$\vec{Z}$ defined with:

$$\varepsilon_0\vec{E}_1=\nabla\times\nabla\times\vec{Z}$$

Derivation of this solution using **Green's functions:**

Solution obtained from: .

$$\nabla^2\vec{Z}-\varepsilon_0\mu_0\frac{\partial^2\vec{Z}}{\partial t^2}=-\varepsilon_1\vec{E}_0$$

$$\text{with}:\ \vec{E}_0(\vec{r}',t')=\vec{E}_{0m}\exp\left[i(\vec{k}_{\vec{r}}\bullet\vec{r}'-\omega_0 t')\right]$$

This is an equation of the form:

$$\nabla^2\Phi-\varepsilon_0\mu_0\frac{\partial^2\Phi}{\partial t^2}=-A$$

use $\Phi=\Psi(x,y,z).e^{i\omega t'}$; $c^2=1/(\varepsilon_0\mu_0)$ and $k=\omega/c$

Now do the derivation to time:

$$\nabla^2\Psi+k^2\Psi=-A' \;;\; \text{with } A'=A.e^{i\omega t'}$$

Solution using Green's theorem:
(with surface S encloses volume V):

$$\iiint_V\left[\Psi_2.\nabla^2\Psi_1-\Psi_1\nabla^2\Psi_2\right]dv=\oiint_S\left[\Psi_2.\frac{\partial\Psi_1}{\partial n}-\Psi_1.\frac{\partial\Psi_2}{\partial n}\right]dS$$

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Appendix B: scattering field

Solution using **Green's Theorem**: Surface S encloses volume V

$$\nabla^2\Psi + k^2\Psi = -A' \;;\; \text{with } A' = Ae^{i\omega t'}$$

$$\iiint_V \left[\Psi_2.\nabla^2\Psi_1 - \Psi_1\nabla^2\Psi_2\right]dv = \oiint_S \left[\Psi_2.\frac{\partial\Psi_1}{\partial n} - \Psi_1.\frac{\partial\Psi_2}{\partial n}\right]dS \quad (1)$$

Green's Theorem follows from **Gauss** Law:

$$\iiint_V \nabla \bullet \vec{X}\, dv = \oiint_S \vec{X} \bullet \vec{n}\, dS$$

upon substituting:

$$\vec{X} = \psi\nabla\varphi \;;\; \vec{X} \bullet \vec{n} = \psi\frac{\partial\varphi}{\partial n} \;;\; \nabla \bullet \vec{X} = \nabla\psi.\nabla\varphi + \psi\nabla^2\varphi$$

Now substitute:

$$\vec{X} = \Psi_2.\nabla\Psi_1 - \Psi_1.\nabla\Psi_2$$

set $\Psi_1 = \Psi$ and try $\Psi_2 = \frac{e^{-ikr}}{r}$ (spherical wave: $\nabla^2\Psi_2 + k^2\Psi_2 = 0 \rightarrow \nabla^2\Psi_2 = -k^2\Psi_2$)

Left side of (1) : $\displaystyle \iiint_V \left[\frac{e^{-ikr}}{r}(-k^2\Psi - A') + k^2\Psi\frac{e^{-ikr}}{r}\right]dv = \iiint_V \left[-A'\frac{e^{-ikr}}{r}\right]dv$

Appendix B: scattering field

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$$\nabla^2\Psi + k^2\Psi = -A' e^{i\omega t} \quad ; \quad \text{with } A' = A e^{-i\omega t}$$

Solution using Green's theorem: Surface S encloses volume V

$$\iiint_V \left[\Psi_2 . \nabla^2\Psi_1 - \Psi_1 \nabla^2 \Psi_2\right] dv = \oiint_S \left[\Psi_2 . \frac{\partial \Psi_1}{\partial n} - \Psi_1 . \frac{\partial \Psi_2}{\partial n}\right] dS \quad (1)$$

$$\text{We did}: \quad \Psi_1 = \Psi \quad \text{and} \quad \Psi_2 = \frac{e^{-ikr}}{r} \quad (\text{spherical wave}: \nabla^2\Psi_2 + k^2\Psi_2 = 0)$$

$$\text{Left side of (1)}: \iiint_V \left[\frac{e^{-ikr}}{r}(-k^2\Psi - A') + k^2\Psi \frac{e^{-ikr}}{r}\right] dv = \iiint_V \left[-A' \frac{e^{-ikr}}{r}\right] dv$$

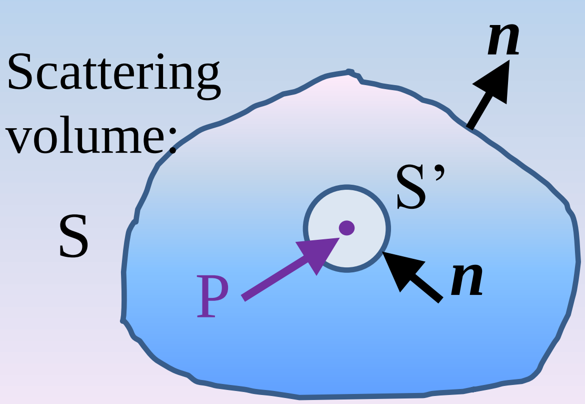
Now: Right side of (1):

P = detection point, to be excluded from *volume* integration

Take spherical volume S' around P and let S' shrink to zero

$$\oiint_S \rightarrow \oiint_S + \oiint_{S'}$$

$$\text{and with} \quad S: \frac{\partial}{\partial n} = \frac{\partial}{\partial r} \quad ; \quad S': \frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$$



$$\oiint_{S'} = \oiint_{S'} \left[\frac{e^{-ikr}}{r} . \frac{-\partial \Psi}{\partial r} - \Psi\left(\frac{e^{-ikr}}{r^2} + . \frac{ike^{-ikr}}{r}\right)\right] dS$$

FdM

Appendix B: scattering field

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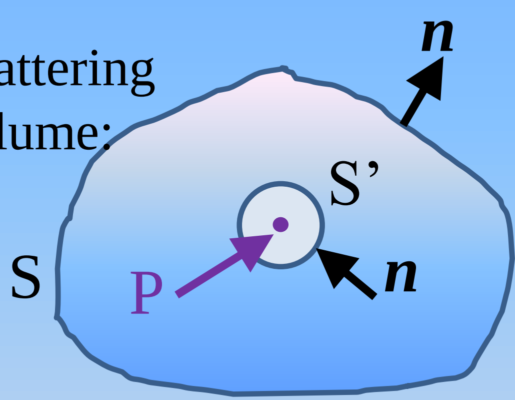
Left side of (1) : $\iiint_V \left[\frac{e^{-ikr}}{r}(-k^2\Psi - A') + k^2\Psi \frac{e^{-ikr}}{r} \right] dv = \iiint_V \left[-A' \frac{e^{-ikr}}{r} \right] dv$

Right side of (1):

P = detection point, to be excluded from *volume* integration

Take spherical volume *S'* around P and let *S'* shrink to zero.

Scattering volume:



$S: \quad \frac{\partial}{\partial n} = \frac{\partial}{\partial r}$;

$S': \quad \frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$

$$\oiint_{S'} = \oiint_{S'} \left[\frac{e^{-ikr}}{r} . \frac{-\partial\Psi}{\partial r} - \Psi(\frac{e^{-ikr}}{r^2} + . \frac{ike^{-ikr}}{r}) \right] dS$$

On *S'* : $dS = r^2.d\Omega$ ($d\Omega$ is integration element over solid angle)

Integration over *S'*, followed by limit $r \rightarrow 0$:

$$\oiint_{S'} = \oiint_{\Omega} \left[\frac{e^{-ikr}}{r} . \frac{-\partial\Psi}{\partial r} - \Psi(\frac{e^{-ikr}}{r^2} + . \frac{ike^{-ikr}}{r}) \right] r^2 d\Omega$$

$\rightarrow 4\pi.\Psi_P$

Left *and* right sides:

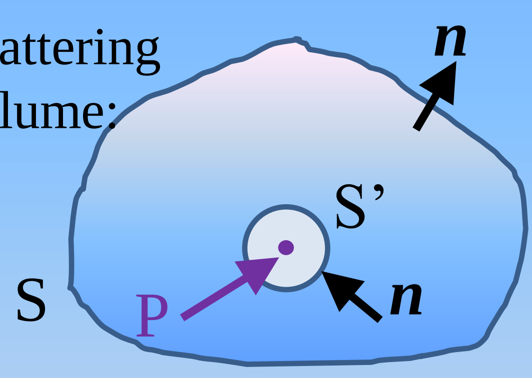
$$\Psi_P = \frac{1}{4\pi} \oiint_S \left[\frac{e^{-ikr}}{r} . \frac{\partial\Psi}{\partial n} - \Psi e^{-ikr} \frac{\partial}{\partial n}(\frac{1}{r}) + \frac{ike^{-ikr}}{r} \Psi \frac{\partial r}{\partial n} \right] dS + \iiint_V \frac{A' e^{-ikr}}{4\pi r} dV$$

Appendix B: scattering field

Result:

$$\Psi_P = \frac{1}{4\pi}\oiint_S \left[\frac{e^{-ikr}}{r} . \frac{\partial \Psi}{\partial n} - \Psi e^{-ikr} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{ike^{-ikr}}{r} \Psi \frac{\partial r}{\partial n} \right] dS + \iiint_V \frac{A' e^{-ikr}}{4\pi r} dV$$

Scattering volume:



Suppose S is very large (>> scattering volume)

and A' is limited in space (finite scattering volume) →

Contribution of S will be = 0 (due to time retardation).

Final result:

$$\Psi_P = \iiint_V \frac{A' e^{-ikr}}{4\pi r} dV$$

Now go back to: $\boldsymbol{\varepsilon_0}$ $\boldsymbol{E_1}$ and $\boldsymbol{Z}$

Appendix B: scattering field

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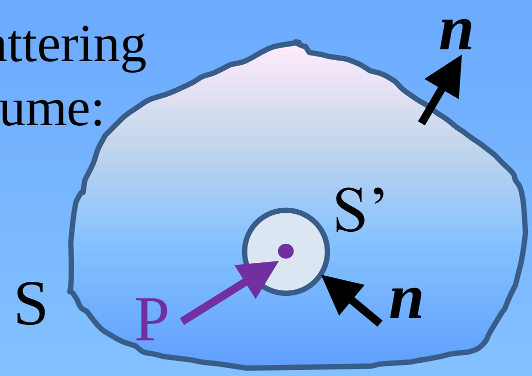
Scattering volume:



$$\Psi_P = \iiint_V \frac{A' e^{-ikr}}{4\pi r} dV$$

We now have solved:

using $\Phi = \Psi(xyz).e^{i\omega t}$

$$\nabla^2 \Phi - \varepsilon_0 \mu_0 \frac{\partial^2 \Phi}{\partial t^2} = -A'$$

$\boldsymbol{Z}$ was represented by Φ
and $\boldsymbol{E_0}$ by $A' = A.\exp(-i\omega t')$

$$\nabla^2 \vec{\boldsymbol{Z}} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{\boldsymbol{Z}}}{\partial t^2} = -\varepsilon_1 \vec{\boldsymbol{E}}_0$$

$$\text{with : } \vec{\boldsymbol{E}}_0(\vec{\boldsymbol{r}}', t') = \vec{\boldsymbol{E}}_{0m} \exp\left[i(\boldsymbol{k}_{\vec{r}} \bullet \vec{\boldsymbol{r}}' - \omega_0 t')\right] : \ t' = \text{retarded time}$$

Now include time retardation:

$$\vec{\boldsymbol{Z}}(\vec{\boldsymbol{R}}, t) = \frac{1}{4\pi} \iiint d^3\vec{\boldsymbol{r}}' \frac{\varepsilon_1 \vec{\boldsymbol{E}}_0(\vec{\boldsymbol{r}}', t')}{\left|\vec{\boldsymbol{R}} - \vec{\boldsymbol{r}}'\right|}$$

$\vec{\boldsymbol{Z}}$ was defined with :

$$\vec{\boldsymbol{D}}_1 \ (= \varepsilon_0 \vec{\boldsymbol{E}}_1) = \nabla \times \nabla \times \vec{\boldsymbol{Z}}$$

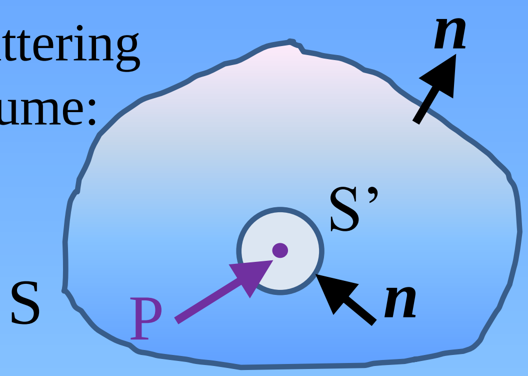
Final result:

$$\vec{\boldsymbol{E}}_1 = \frac{1}{4\pi\varepsilon_0} \iiint d^3\vec{\boldsymbol{r}}' . \vec{\boldsymbol{k}} \times \vec{\boldsymbol{k}} \times \frac{\varepsilon_1 \vec{\boldsymbol{E}}_0(\vec{\boldsymbol{r}}', t')}{\left|\vec{\boldsymbol{R}} - \vec{\boldsymbol{r}}'\right|}$$

Q.E.D.

Appendix B: scattering field

Scattering volume:



Final result:

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0} \iiint d^3\vec{r}\,' \,.\, \vec{k} \times \vec{k} \times \frac{\varepsilon_1 \vec{E}_0(\vec{r}\,', t')}{\left|\vec{R} - \vec{r}\,'\right|}$$

Mie (and others) have calculated this integral expression for various geometries and dielectric constant distributions.

In case the incident light has a beam shape
with symmetry around the symmetry axis of the beam,
an approach with Bessel/Legendre functions is feasible.
but this goes beyond the goal of this presentation.

End of this presentation

Thank you for your attention