Laplace Potential Distribution and Earnshaw's Theorem

Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

Laplace and Earnshaw

1. **Electric Field** equations:
   – Gauss' Law and Potential Gradient Law
2. **Laplace and Poisson**: derivation
3. **Laplace and Poisson in 1 dimension**
4. Charge-free space: **Earnshaw's Theorem**
   – Finite-Elements method for Potential Distribution
5. **Laplace and Poisson in 2 and 3 dimensions**

Electric Field Equations

Gauss: integral formulation:

$$\oint\!\!\!\!\oint_S \boldsymbol{E} \bullet \boldsymbol{dS} = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV$$

Id. differential formulation:

$$\nabla \bullet \boldsymbol{E}(x, y, z) = \frac{\rho(x, y, z)}{\varepsilon_0}$$

$$\left\{ \vec{\boldsymbol{e}}_{\boldsymbol{x}} \frac{\partial}{\partial x} + \ldots \right\} \bullet \left\{ E_x \vec{\boldsymbol{e}}_{\boldsymbol{x}} + \ldots \right\} = \frac{\partial E_x}{\partial x} + \ldots = \frac{\rho}{\varepsilon_0}$$

Potential: integral formulation:

$$V_B - V_A = -\int_A^B \boldsymbol{E} \bullet \boldsymbol{dl}$$

Id. differential formulation:

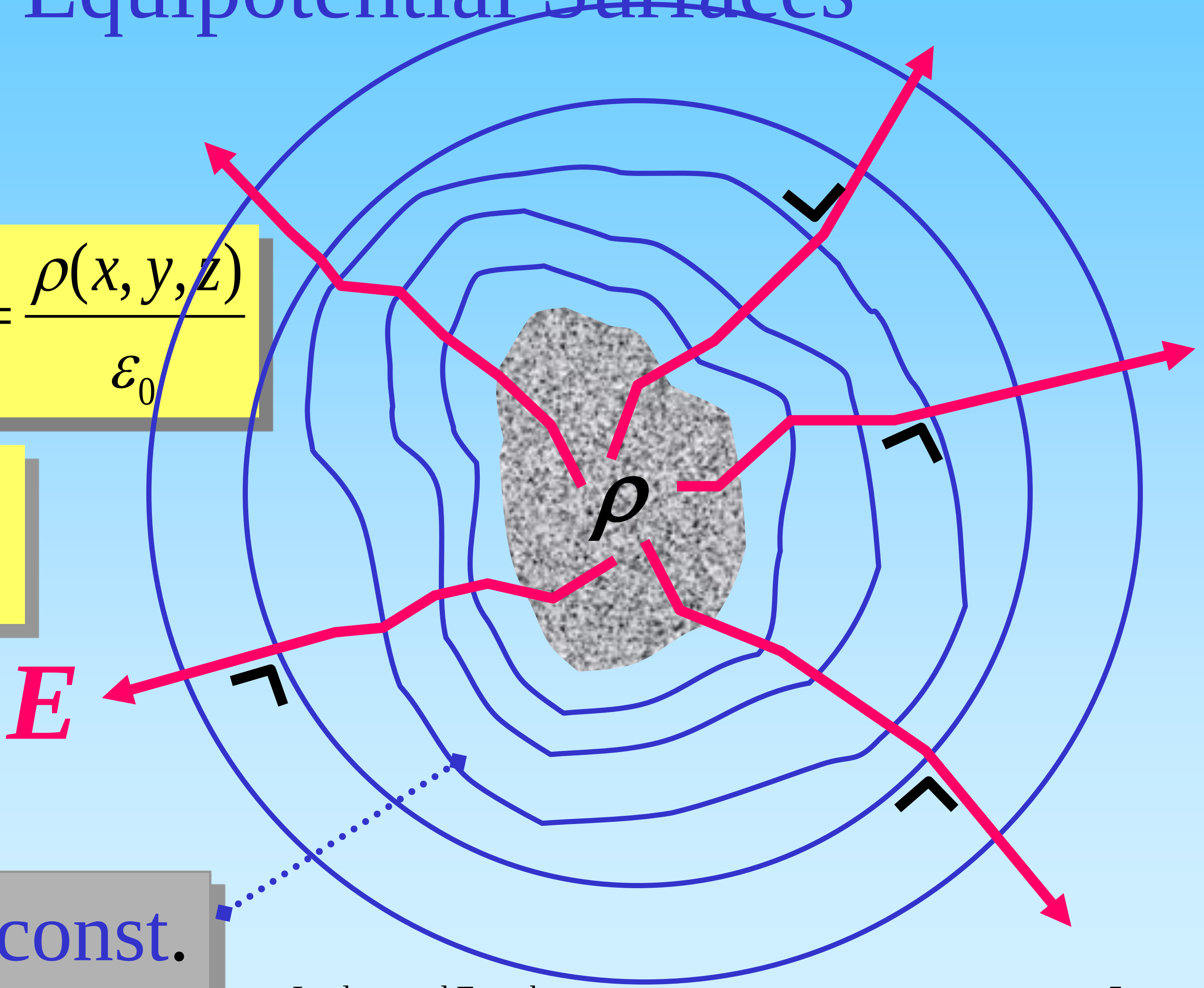
$$\boldsymbol{E} = -\nabla V$$

$$\left\{ \vec{\boldsymbol{e}}_{\boldsymbol{x}} \frac{\partial}{\partial x} + \ldots \right\} V(x, y, z) = \vec{\boldsymbol{e}}_{\boldsymbol{x}} \frac{\partial V}{\partial x} + \ldots = -\vec{\boldsymbol{e}}_{\boldsymbol{x}} E_x - \ldots$$

Electric Field Lines and Equipotential Surfaces

$$\nabla \bullet \boldsymbol{E}(x,y,z) = \frac{\rho(x,y,z)}{\varepsilon_0}$$

$$\boldsymbol{E} = -\nabla V$$



V = const.

Laplace and Poisson: derivation

Gauss: $\nabla \bullet \boldsymbol{E}(x, y, z) = \dfrac{\rho(x, y, z)}{\varepsilon_0}$

Potential: $\boldsymbol{E} = -\nabla V$

Laplace / Poisson : $\nabla \bullet \boldsymbol{E} = -\nabla \bullet \nabla V = -\nabla^2 V = \dfrac{\rho}{\varepsilon_0}$

$$\vec{\mathbf{e}}_{\mathbf{x}} \frac{\partial}{\partial x} \bullet \vec{\mathbf{e}}_{\mathbf{x}} \frac{\partial V}{\partial x} + \ldots = -\frac{\rho}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0}$$

$\rho = 0$: free space (Laplace)

$\rho \neq 0$: materials (Poisson)

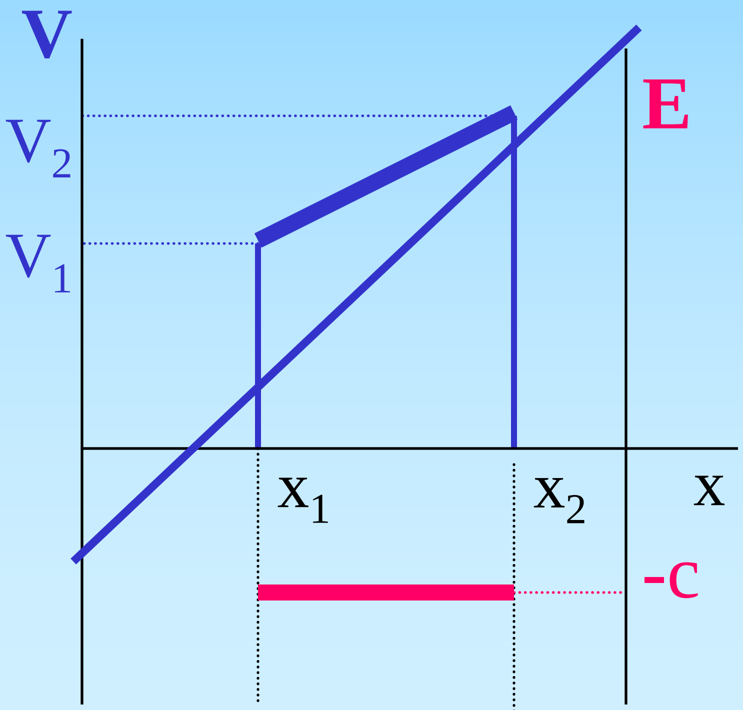
Laplace and Poisson in 1 dimension

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho}{\varepsilon_0}$$

$\rho = 0$: free space (Laplace)
$\rho \neq 0$: materials (Poisson)

Calculate *V(x)* for $\boldsymbol{\rho} = \mathbf{0}$ by integration of Laplace equation



$$\frac{d^2V}{dx^2} = 0 \quad \Rightarrow \frac{dV}{dx} = c = -E_x$$

$$\Rightarrow V(x) = cx + c'$$

<u>Boundary conditions</u>:
V_1 at x_1 and V_2 at x_2 :

$$V(x) = V_1 + \frac{x - x_1}{x_2 - x_1}(V_2 - V_1)$$

Laplace and Poisson in 1 dimension

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho}{\varepsilon_0}$$

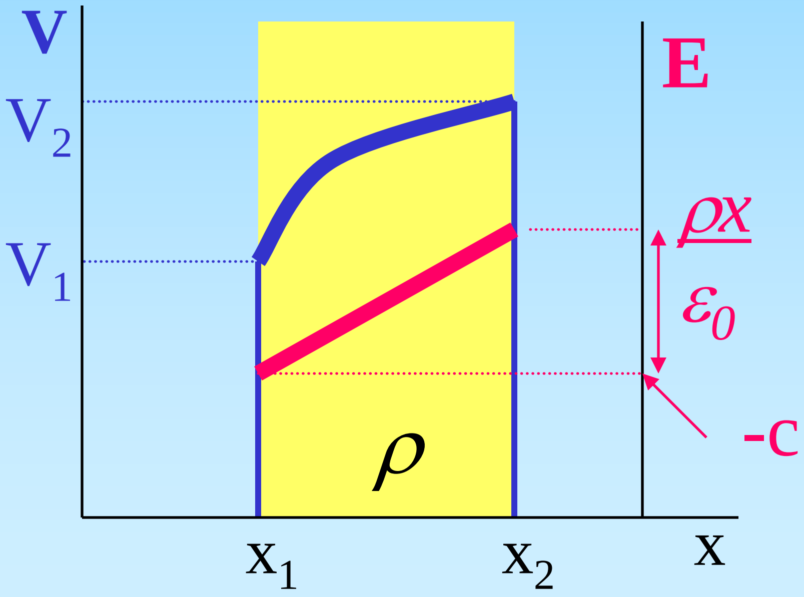
$\rho = 0$: free space (Laplace)

$\rho \neq 0$: materials (Poisson)

Calculate *V(x)* by integration of Poisson's equation….

$$\Rightarrow \frac{dV}{dx} = c - \frac{\rho}{\varepsilon_0} x = -E_x$$

$$\Rightarrow V(x) = cx - \tfrac{1}{2}\frac{\rho}{\varepsilon_0} x^2 + c'$$



Assume ρ =const.:

<u>Boundary conditions</u> at x_1 and x_2

$\Rightarrow$ Parabolic behaviour

Laplace and Poisson in 1 dimension

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho}{\varepsilon_0}$$

$\rho = 0$: free space (Laplace)

$\rho \neq 0$: materials (Poisson)

$$\Rightarrow V(x) = cx - \tfrac{1}{2}\frac{\rho}{\varepsilon_0}x^2 + c'$$

Assume ρ =constant:

Boundary conditions at x_1 and x_2

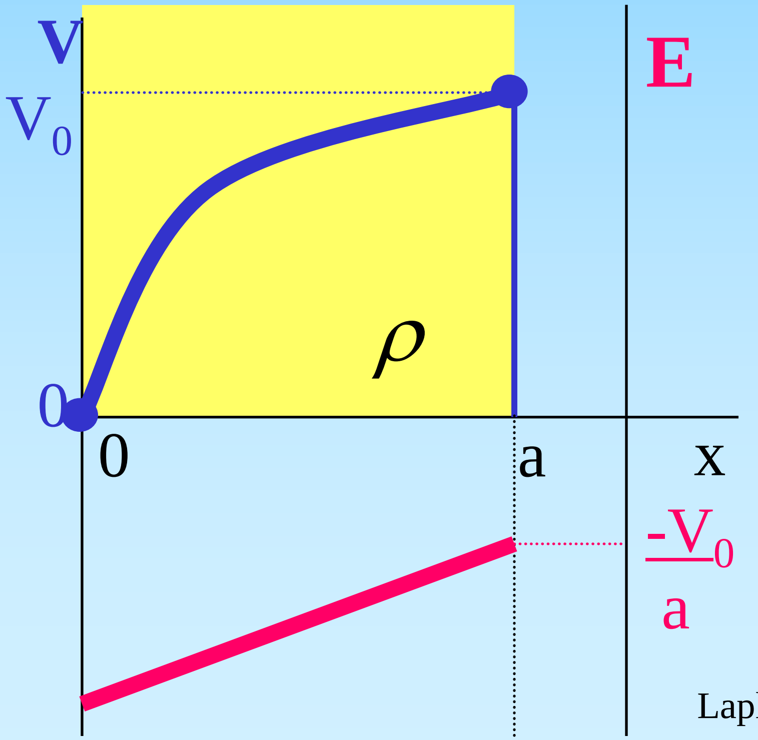
Special case:

$x_1=0$; $V_1=0$ and $x_2= a$; $V_2 = V_0$

Calculate $V(x)$ and $E(x)$

$$V(x) = -\frac{\rho x^2}{2\varepsilon_0} + \frac{\rho a x}{2\varepsilon_0} + \frac{V_0 x}{a}$$

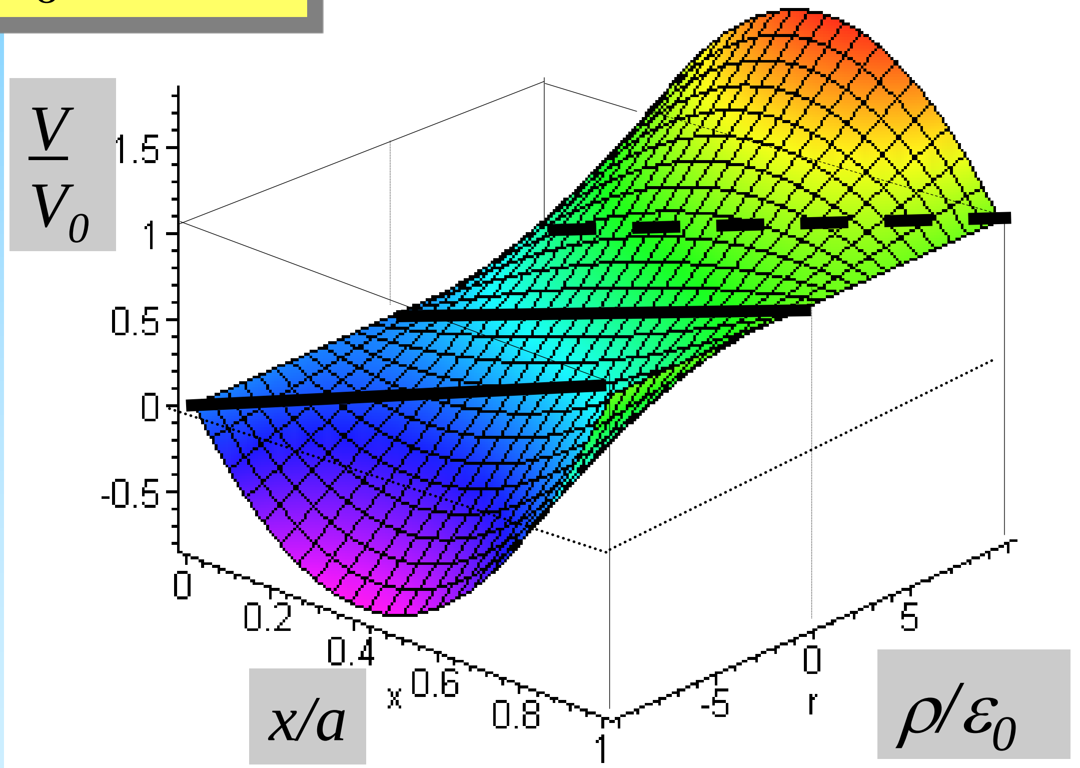
$$E(x) = -\left[\frac{\rho}{\varepsilon_0}(-x+a) + \frac{V_0}{a}\right]$$



Laplace and Poisson in 1 dimension

$$V(x) = -\frac{\rho x^2}{2\varepsilon_0} + \frac{\rho a x}{2\varepsilon_0} + \frac{V_0 x}{a}$$

Assume ρ =const.:
<u>Boundary conditions:</u>
at x_1 and x_2
<u>**Special case**</u>:
$x_1=0\ ;\ V_1=0$ and
$x_2=a\ ;\ V_2=V_0$



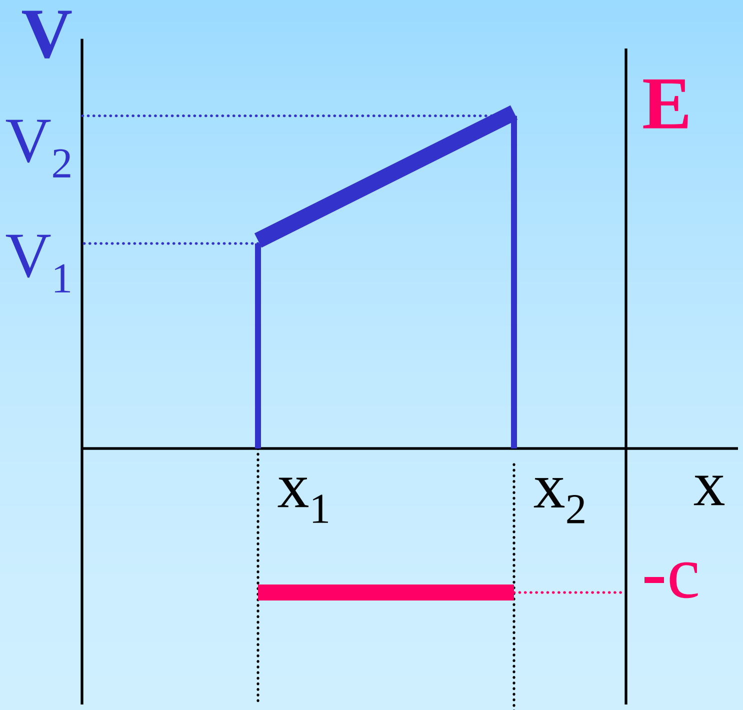
Laplace in 1 dimension

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho}{\varepsilon_0}$$

$\rho = 0$: free space (Laplace)

$\rho \neq 0$: materials (Poisson)



$$\frac{d^2 V}{dx^2} = 0 \Rightarrow \frac{dV}{dx} = c = -E_x$$

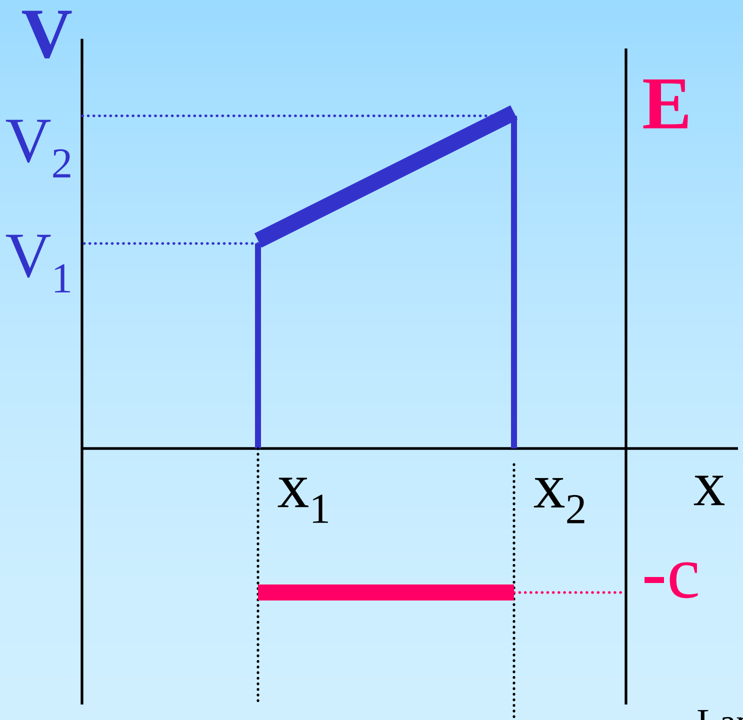
$$\Rightarrow V(x) = cx + c'$$

Boundary conditions at x_1 and x_2 :

$$V(x) = V_1 + \frac{x - x_1}{x_2 - x_1}(V_2 - V_1)$$

Laplace in 1 dimension: Earnshaw

$$\frac{d^2V}{dx^2} = 0 \quad \Rightarrow \frac{dV}{dx} = c = -E_x \quad \Rightarrow V(x) = cx + c'$$



Earnshaw:

If no free charge present, then: Potential has no local maxima or minima.

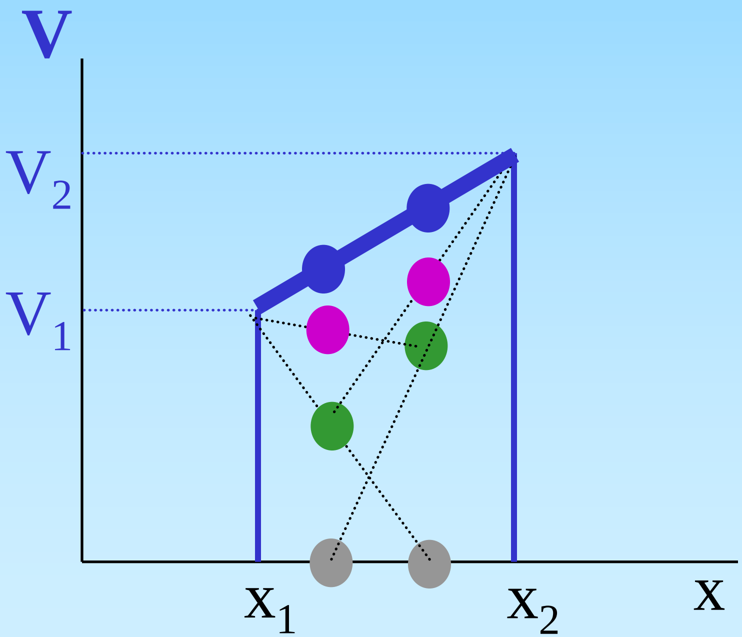
Consequences:

1. V is linear function of position
2. V at each point is always in between neighbours

Laplace in 1 dimension: Earnshaw

Earnshaw:

If no free charge present, then: Potential has no local maxima or minima.



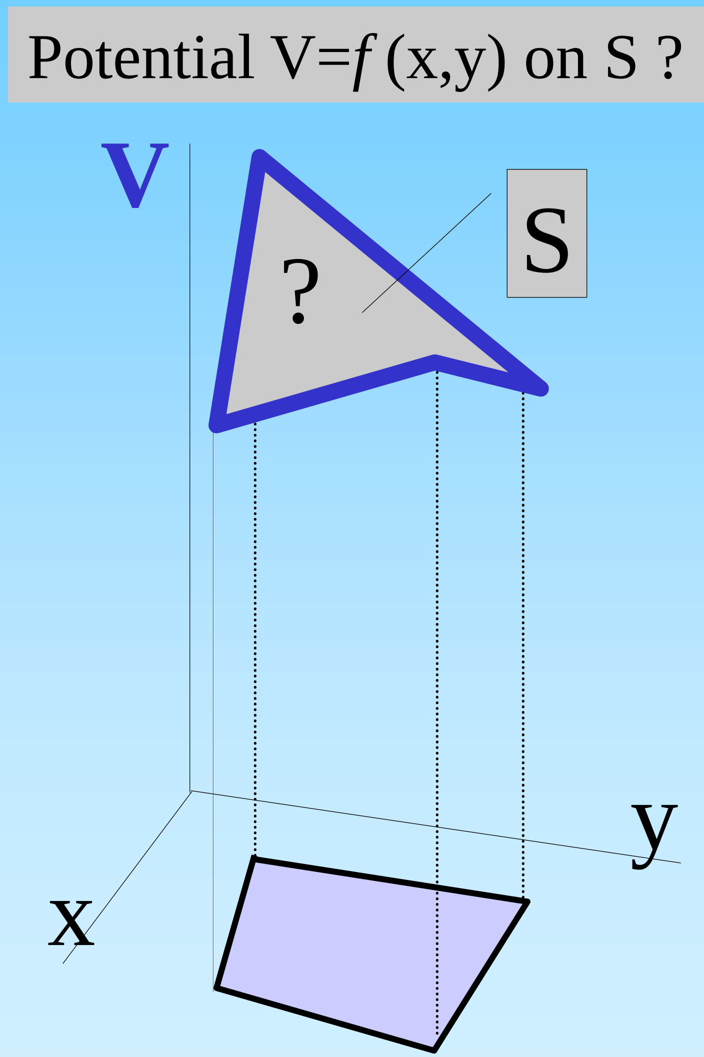
Consequences:

1. V is linear function of position
2. V at each point is always in between neighbours

Numerical method for calculating potentials between boundaries:

1. Start with zero potential between boundaries
2. Take averages between neighbours
3. Repeat and repeat and ....

Laplace in 2 dimensions: Earnshaw



Earnshaw:

If no free charge present, then: Potential has no local maxima or minima.

Solution of Laplace $\nabla^2 V = 0$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] V(x, y) = 0$$

will depend on boundaries.

"Partial differential equation"

Laplace / Poisson in 3 dimensions

Spatial charge density: ρ = *f* (x,y,z)

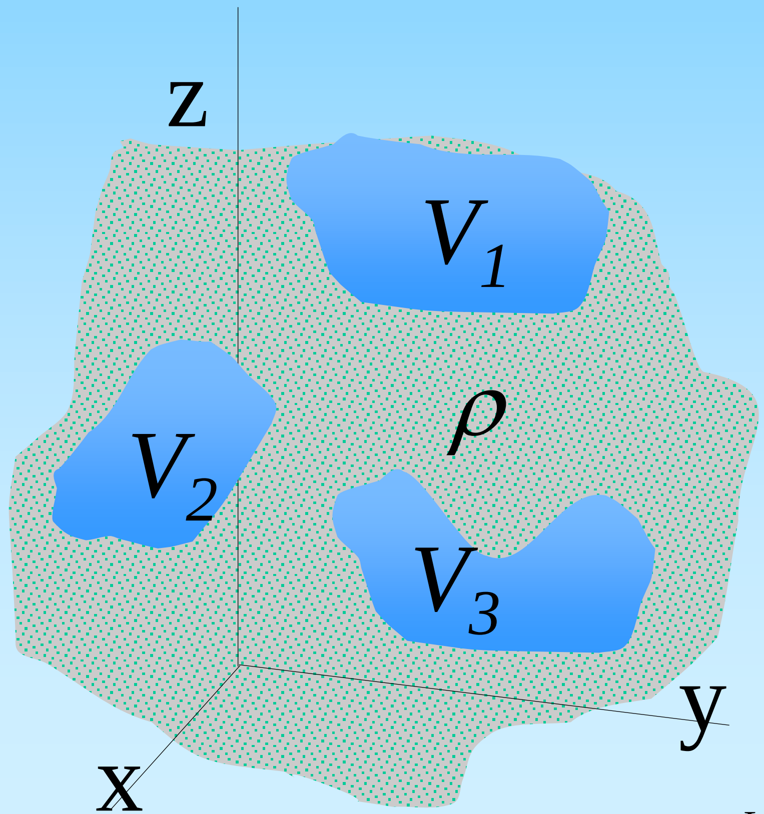
Potential *V* = *f* (x,y,z) ?

Boundary conditions:
V_1 , V_2 and V_3 = *f* (x,y,z)

Solution of Laplace/Poisson:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]V(x,y)=-\frac{\rho}{\varepsilon_0}$$

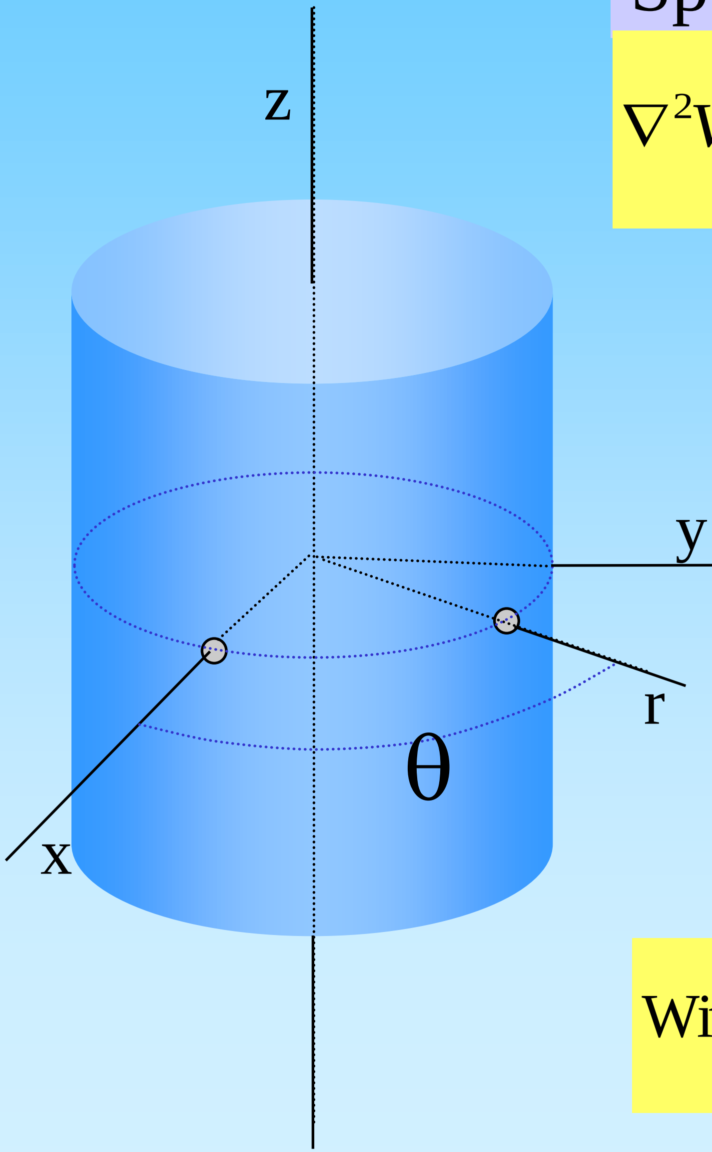
will depend on boundaries.



4-D plot needed !?

Special cases:
- cylindrical geometry
- spherical geometry

Laplace / Poisson in 3 dimensions



Special case 1: Cylindrical geometry

$$\nabla^2 V = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right] V = -\frac{\rho}{\varepsilon_0}$$

If *r* – dependence only:

ρ and boundaries will be *f* (*r*).

Thus: *V* will be *f* (*r*) only

<u>Example</u>: $V=V_1$ at r_1 and V_2 at r_2 , and $\rho=0$

Calculate *V* = *V(r)*

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0 \quad \Rightarrow \quad r\frac{dV}{dr} = c \quad \Rightarrow \quad \frac{dV}{dr} = \frac{c}{r}$$

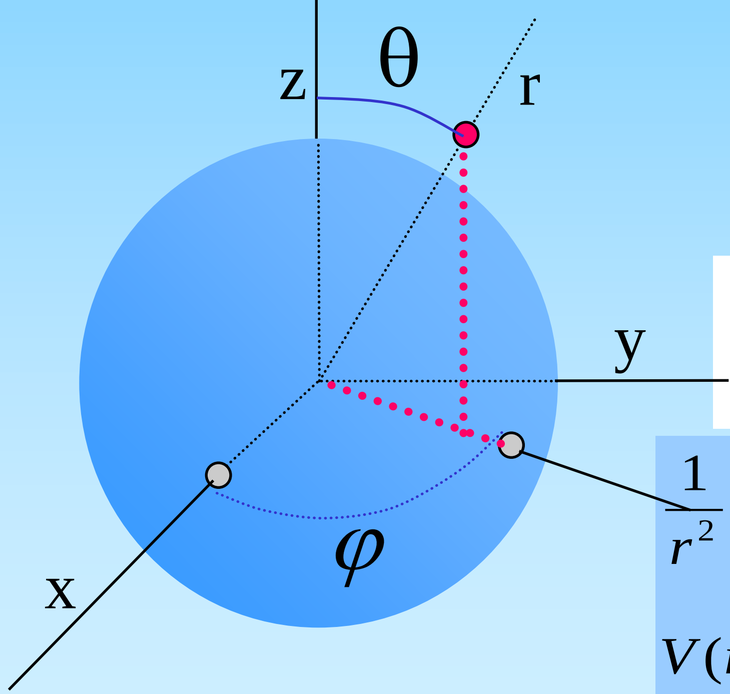
$$V(r) = c.\ln r + c' \qquad \text{(c and c': const.)}$$

$$\text{With boundaries}: \quad V(r) = V_1 + (V_2 - V_1)\frac{\ln r - \ln r_1}{\ln r_2 - \ln r_1}$$

Laplace / Poisson in 3 dimensions

Special case 2: Spherical geometry

$$\nabla^2 V = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]V = -\frac{\rho}{\varepsilon_0}$$



If r – dependence only:

$\boldsymbol{\rho}$ and boundaries will be $f(r)$.

Thus: V will be $f(r)$ only

<u>Example</u>: $V=V_1$ at r_1 and V_2 at r_2, and $\boldsymbol{\rho}=0$

Calculate $V = V(r)$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \quad\Rightarrow\quad r^2\frac{dV}{dr} = c \quad\Rightarrow\quad \frac{dV}{dr} = \frac{c}{r^2}$$

$$V(r) = c.\frac{-1}{r} + c'$$ (c and c': cons

the end

With boundaries : $V(r) = V_1 + (V_2 - V_1)\dfrac{1/r - 1/r_1}{1/r_2 - 1/r_1}$