First-year course on

Electromagnetism

Integral types

Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

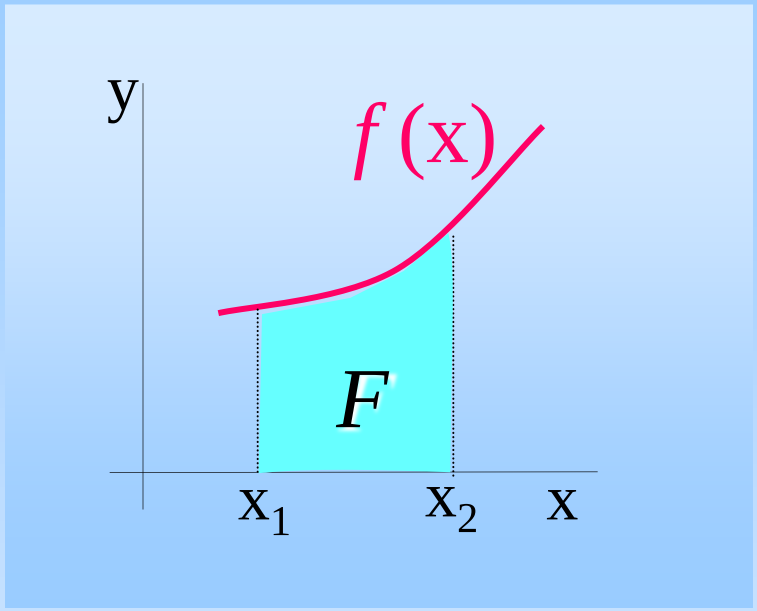
Multidimensional integrations

Integration of charges and currents
over lines, planes and volumes

1. Line integrals (scalar, vectorial) +example
2. Surface-integrals (scalar, vectorial) + 2 examples
3. Volume-integrals (scalar only) + example

To perform integrations:
always rewrite **charge (*dQ*) and current elements (*dI*)**
into **coordinate elements (*dx*,*dy*,*dz***…or other)
using **charge and current densities: (λ, σ, ρ and *j*).**

Line-integral (scalar) (1)



One-dimensional:

$$F = \int_{x_1}^{x_2} f(x)\, dx$$

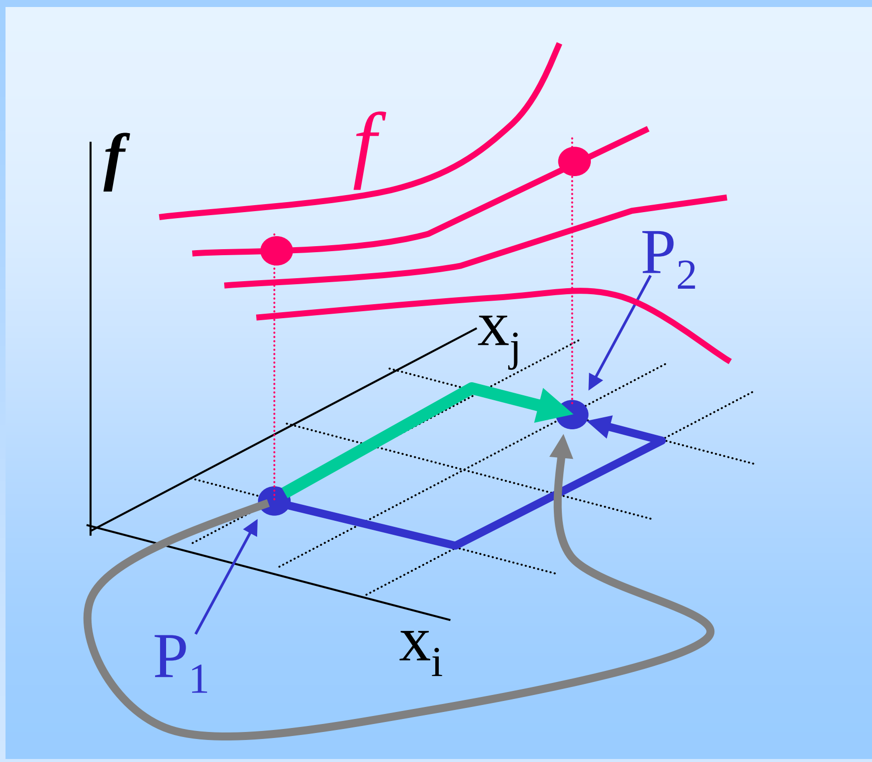
Meaning of F:

F = area under curve

Example: Calculate average temperature *T*(x) between x_1 and x_2

$$< T >_{x_1 \ldots x_2} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} T(x)\, dx$$

Line-integral (scalar) (2)



Multi-dimensional

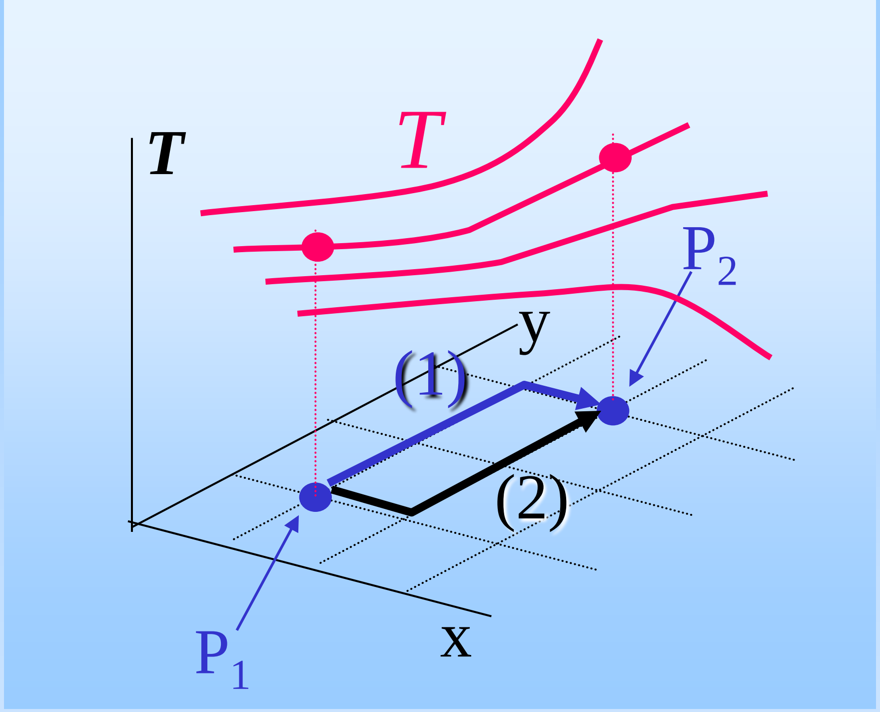
$f = f\,(\ldots\ldots, x_i\,, x_j\,,\ldots.)$

$$F = \int_{P_1}^{P_2} f\,(\ldots, x_i, x_j, \ldots)\, dl$$

Problem:
which integration path ?

In general: Result of integration depends on choice of integration path.

Line-integral (scalar) (3)



Example

T (x,y) = c (2x+y)
P_1 (1,1)
P_2 (2,3)

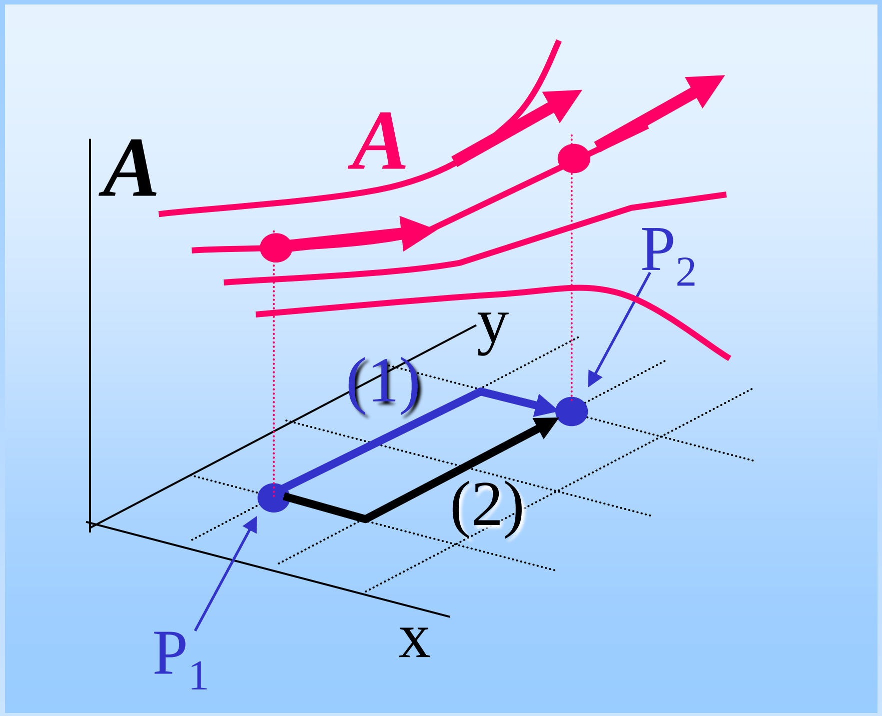
Different paths :
different results;
Calculate line-integrals first
and check below:

(1) $$\int_{P_1}^{P_2} c(2x+y)dl = \int_1^3 c(2.1+y)dy + \int_1^2 c(2x+3)dx = c(6-2+\tfrac{9}{2}-\tfrac{1}{2}) + c(4-1+6-3) = 14c$$

(2) $$\int_{P_1}^{P_2} c(2x+y)dl = \int_1^2 c(2.x+1)dx + \int_1^3 c(2.2+y)dy = c(4-1+2-1) + c(12-4+\tfrac{9}{2}-\tfrac{1}{2}) = 16c$$

<u>Special case: Conservative field: </u>result <u>in</u>dependent of path

Line-integral (vectorial)



Definition:

$$\int_{P_1}^{P_2} \vec{\boldsymbol{A}} \bullet \boldsymbol{d}\vec{\boldsymbol{l}} = \int_{P_1}^{P_2} A.dl.\cos\theta$$

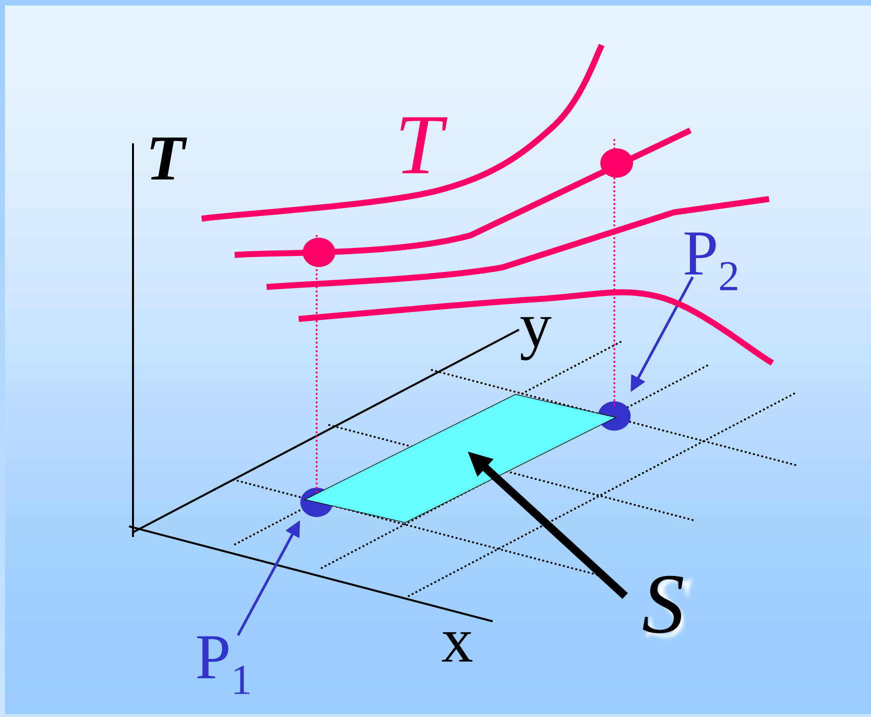
dl along integration path

Example: Work done by force:

$$W = \int_{P_1}^{P_2} \vec{\boldsymbol{F}} \bullet \boldsymbol{d}\vec{\boldsymbol{l}} = \int_{P_1}^{P_2} F.dl.\cos\theta$$

Consequence: if $\boldsymbol{F} \perp \boldsymbol{dl}$: $W = 0$:
(examples: centripetal force; magnetic force)

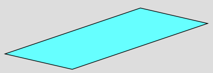
Surface-integral (scalar)



<u>Temperature field:</u>

$T = f\,(x,y) = c(2x+y)$

<u>Problem:</u> determine

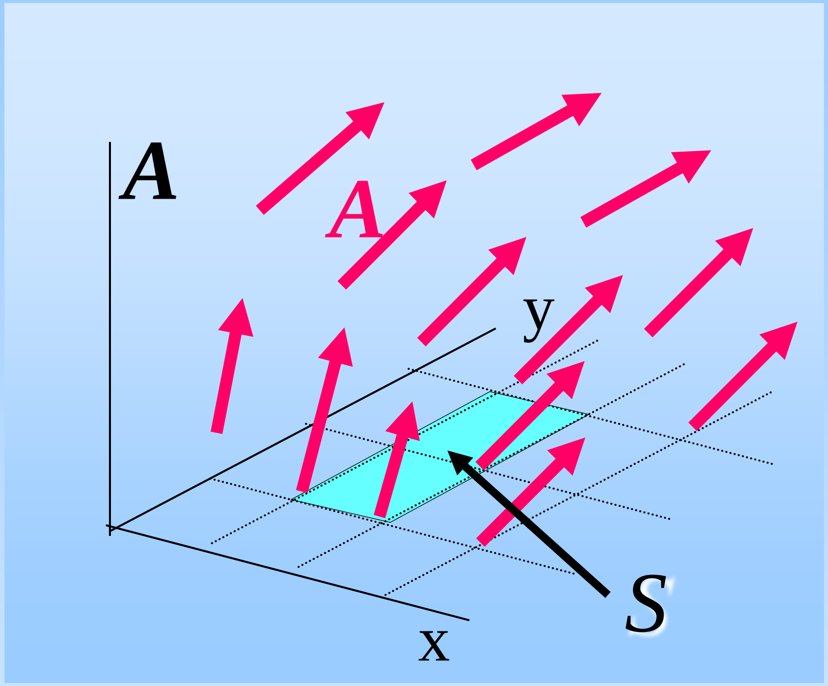
$< T >$ over S; 

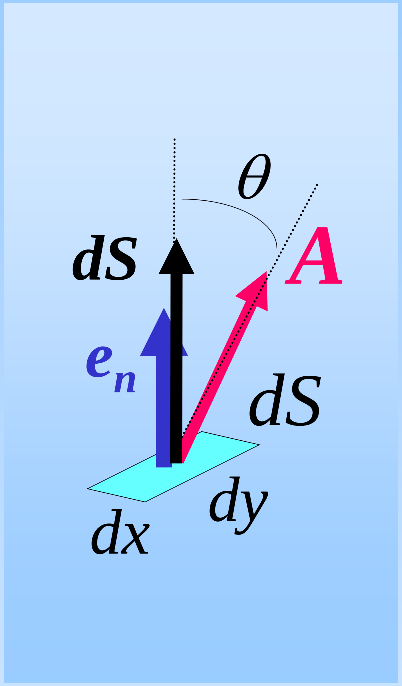
Find formula and calculate:

$$< T >_S = \frac{\iint_S T(x,y)\,dS}{\iint_S dS}$$

$$< T >_S = \frac{\int_{x=1}^{2} \int_{y=1}^{3} c(2x+y)\,dx\,dy}{\int_{x=1}^{2} \int_{y=1}^{3} dx\,dy} = \frac{\int_{x=1}^{2} c(4x+4)\,dx}{1.2} = \frac{10c}{2} = 5c$$

Surface-integral (vectorial)





$\boldsymbol{e_n}$ = normal unit vector, // to $\boldsymbol{dS}$

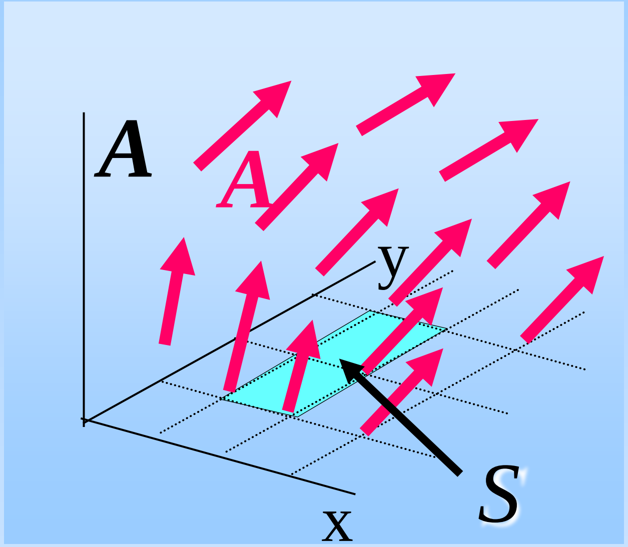
$\boldsymbol{dS} = \boldsymbol{e_n}\, dS$

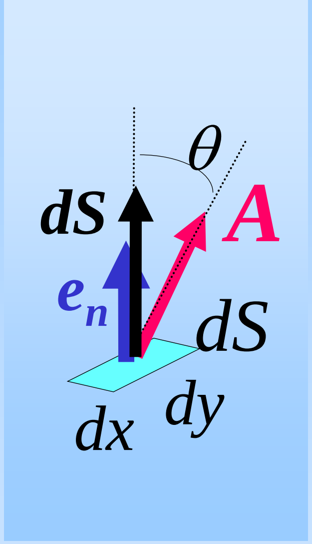
Definition:

$$\iint_S \vec{\boldsymbol{A}} \bullet \boldsymbol{d\vec{S}} = \iint_S \vec{\boldsymbol{A}} \bullet \vec{\boldsymbol{e}}_{\boldsymbol{n}} dS = \int_x \int_y A(x,y).dxdy.\cos\theta$$

Surface Integral (vectorial): Example 1

Suppose: Area S in z=0 plane ; there $\boldsymbol{A}(x,y,0) = x\boldsymbol{e}_x + 2y\boldsymbol{e}_y + 3\boldsymbol{e}_z$





$\boldsymbol{e}_n$ = normal unit vector,
$\boldsymbol{e}_n$ // $\boldsymbol{dS}$

$\boldsymbol{dS} = \boldsymbol{e}_n\, dS$

Calculate surface integral over x = 1..2 and y = 1..3

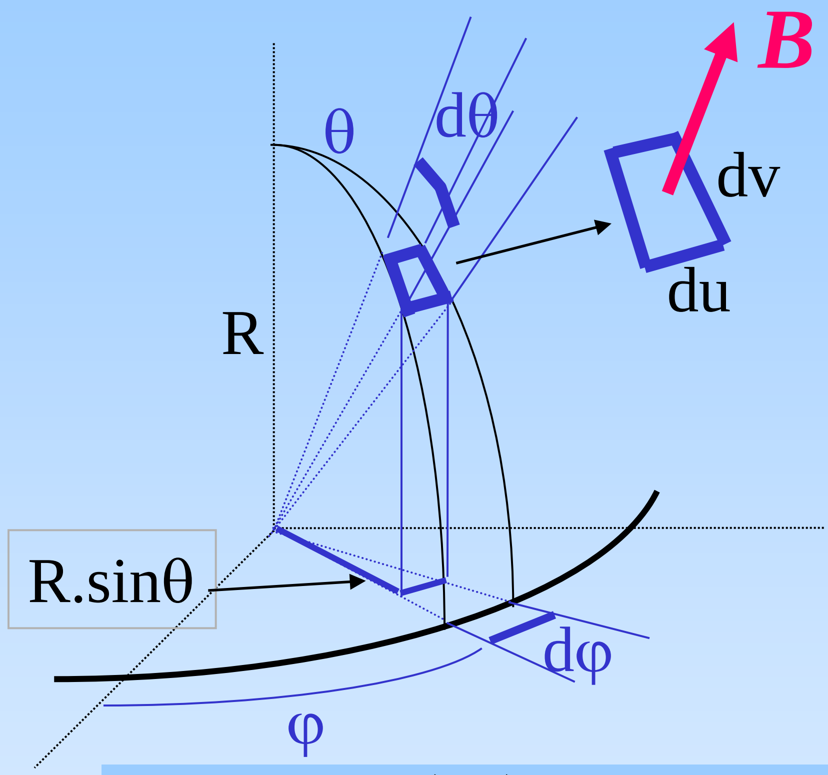
$$\iint_S \boldsymbol{A} \bullet \boldsymbol{dS} = \iint_S \boldsymbol{A} \bullet \boldsymbol{e}_n dS = \int_{x=1}^{2} \int_{y=1}^{3} A(x,y).dxdy.\cos\theta =$$

$$= \int_{x=1}^{2} \int_{y=1}^{3} [x\boldsymbol{e}_x + 2y\boldsymbol{e}_y + 3\boldsymbol{e}_z] \bullet \boldsymbol{e}_z\ dxdy = \int_{x=1}^{2} \int_{y=1}^{3} [0+0+3]dxdy = 3.2.1 = 6$$

Contribution from ⊥-component (// $\boldsymbol{e}_z$) only !

Surface Integral (vectorial): Example 2

Spherical surface element



dA = du.dv = (R.sinθ.dφ).(Rdθ)

Suppose:
$\boldsymbol{B} = r.\sin\theta\,\boldsymbol{e_r} + \cos\varphi\,\boldsymbol{e_\theta} + \tan\theta\,\boldsymbol{e_\varphi}$

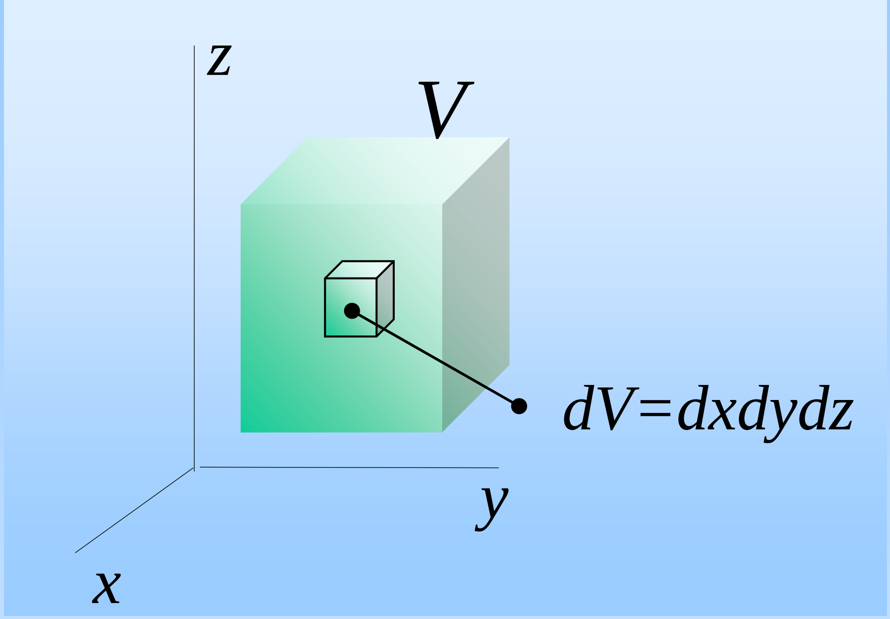
Calculate surface integral of $\boldsymbol{B}$
$\iint \boldsymbol{B} \bullet \boldsymbol{dA}$ over A at radius r
over octant (θ=0.. ½π ; φ=0..½ π)

Normal vector $\boldsymbol{e_n} = \boldsymbol{e_r}$ everywhere !

$$\iint_{area} \boldsymbol{B}\bullet\boldsymbol{dA} = \int_{\theta=0}^{\frac{1}{2}\pi}\int_{\varphi=0}^{\frac{1}{2}\pi}[R.\sin\theta\boldsymbol{e_r} + \cos\varphi\boldsymbol{e_\theta} + \tan\theta\boldsymbol{e_\varphi}]\bullet\boldsymbol{e_n}.\,R^2.\sin\theta.d\theta.d\varphi =$$

$$\int_{\theta=0}^{\frac{1}{2}\pi}\int_{\varphi=0}^{\frac{1}{2}\pi}[R.\sin\theta + 0 + 0].R^2\sin\theta.d\theta.d\varphi = \int_{\theta=0}^{\frac{1}{2}\pi}\int_{\varphi=0}^{\frac{1}{2}\pi} R^3.\sin^2\theta.d\theta.d\varphi = \tfrac{1}{8}\pi^2 R^3.$$

Volume-integral (scalar only)



Block V : limited by points (1,1,1) and (2,4,5)

Example: Charge density:
$\rho = c(3x\text{-}2y+5z)$ [C/m^3]

Problem: determine total charge Q in V

Define charge element dQ in volume element dV : $dQ = \rho .dV$

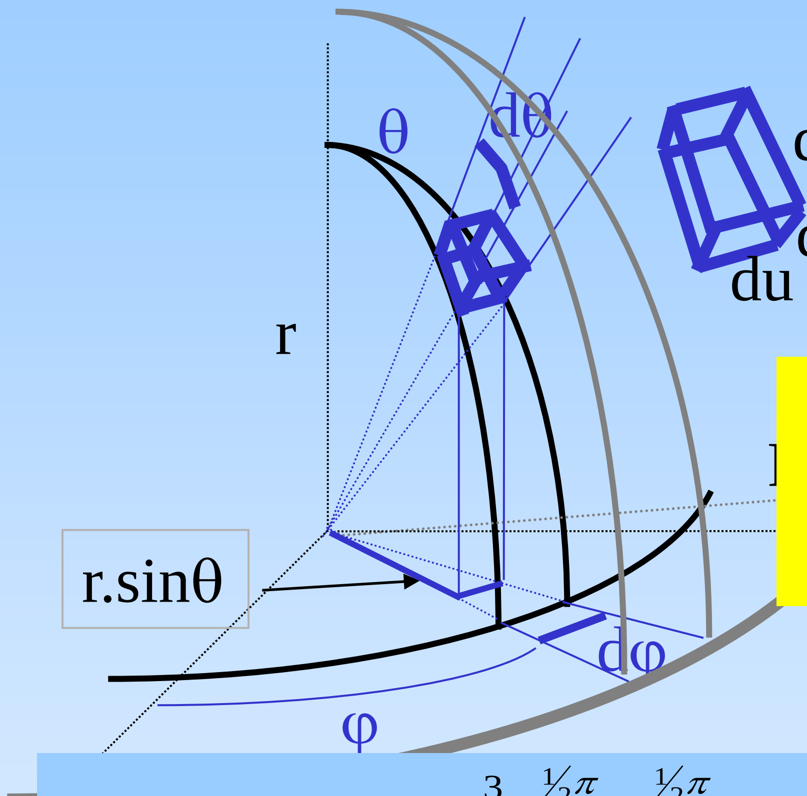
$$Q_V = \iiint_V \rho(x,y,z).dV = \int_{x=1}^{2}\int_{y=1}^{4}\int_{z=1}^{5} c(3x-2y+5z)\, dxdydz = 174.c$$

Volume integral: Example

Spherical volume element

dV = du.dv.dw = (r.sinθ.dφ).(rdθ).dr

0< φ < 2π ; 0< θ < π ; 0 < r < R



Suppose: charge density

$\rho=3ar.\sin\theta.\cos\varphi$ [C/m^3]

the end

charge Q in region:

$<\theta<½\pi$; $-½\pi<\varphi<½\pi$)

$$Q=\iiint_{region}\rho.dv=\int_{r=2}^{3}\int_{\theta=0}^{½\pi}\int_{\varphi=-½\pi}^{½\pi}[3ar.\cos\theta\cos\varphi].r.\sin\theta.d\varphi.r.d\theta.dr=$$

$$3a\int_{2}^{3}r^3dr\int_{0}^{\pi/2}-\cos\theta.d(\cos\theta)\int_{-\pi/2}^{\pi/2}\cos\varphi.d\varphi=3a.\frac{81-16}{4}.\tfrac{1}{2}.2=\frac{195a}{4}$$