Electric Field of charged hollow and solid Spheres

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Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

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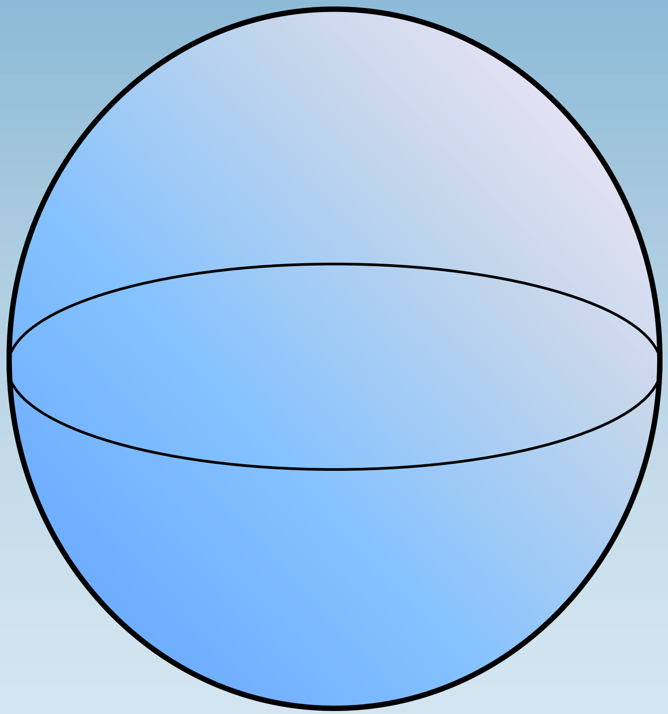
E-field of a charged sphere



Available:

A sphere, radius R, with

- surface charge density σ [C/m^2], or
- volume charge density ρ [C/m^3]

σ and ρ can be f (position)

Question:

Calculate ***E***-field in arbitrary points inside and outside the sphere

E-field of a charged sphere

Contents:

1. A hollow sphere, homogeneously charged (conducting)
2. Idem., non-homogeneously charged
3. A solid sphere, homogeneously charged
4. Idem., non-homogeneously charged

Method: integration over charge elements (radial and angular integrations).

NB. Homogeneously charged spheres can also be calculated in an easy way using the symmetry in Gauss' Law.

E-field of a charged sphere

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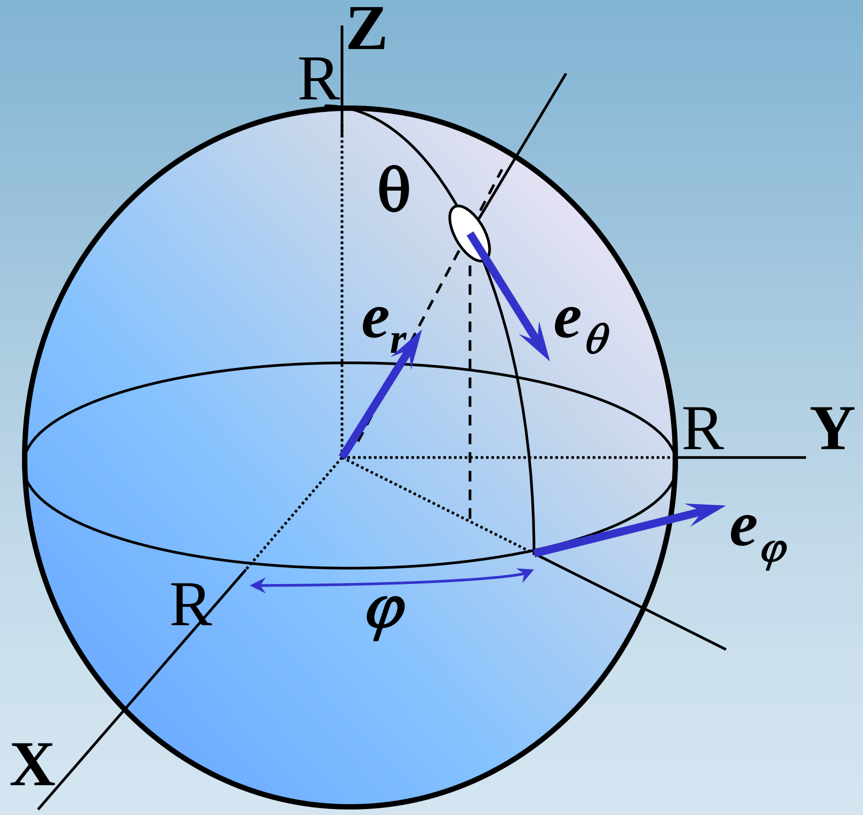
1. Hollow sphere, homogeneously charged



1. Charge distribution:

   (surface charge) σ [C/m^2]

2. Coordinate axes:

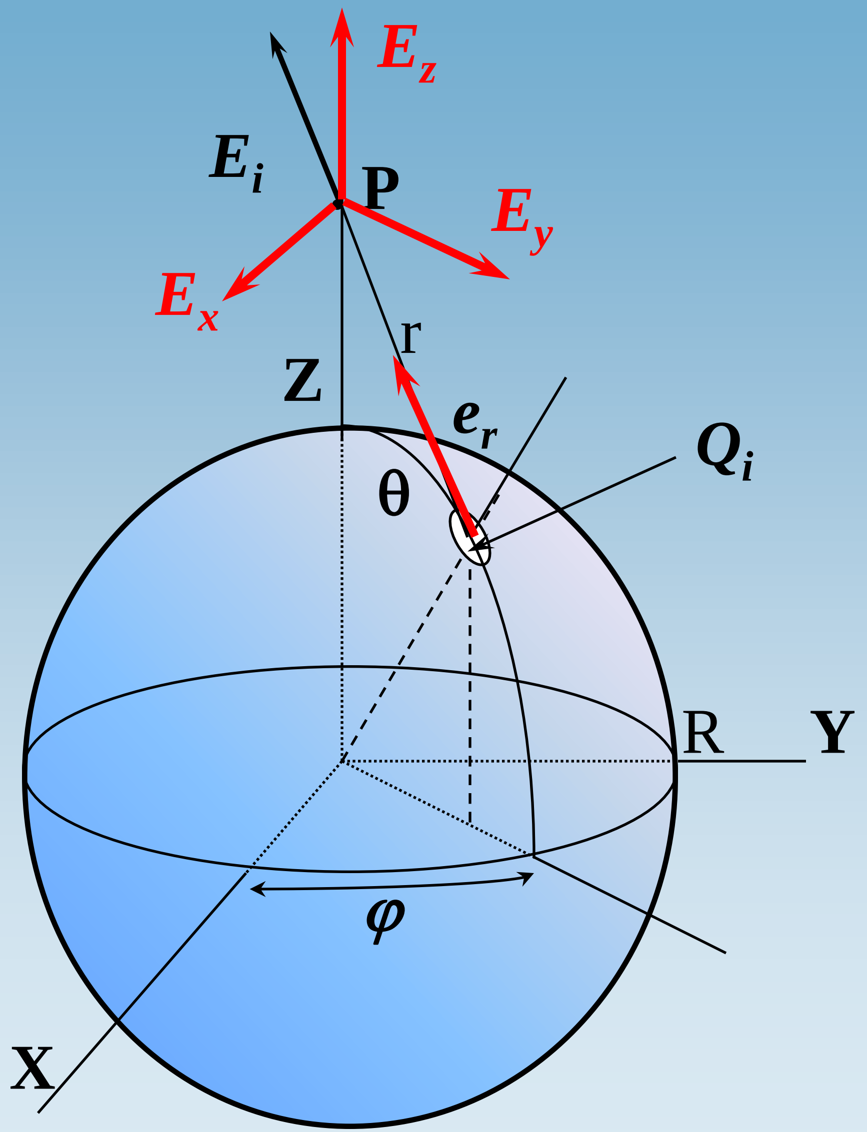
   Z-axis = polar axis

3. Symmetry: spherical

4. Spherical coordinates:

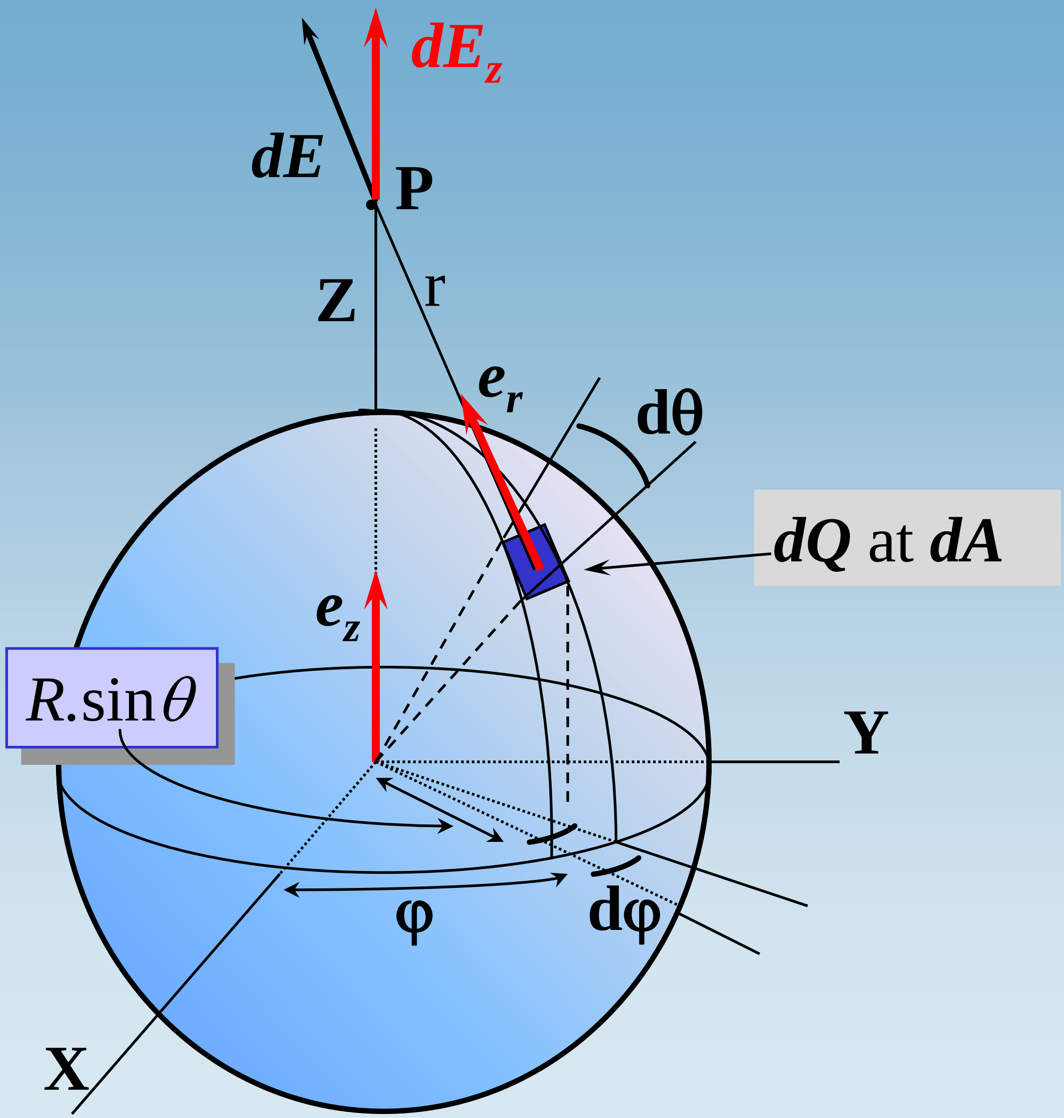
   r, θ, φ

1. Hollow sphere, homogeneously charged



1. XYZ-axes
2. Point P on Z-axis
3. all Q_i's at r_i , θ_i , φ_i contribute $\boldsymbol{E_i}$ to $\boldsymbol{E}$ in P
4. In P: $\boldsymbol{E_{i,x}}$, $\boldsymbol{E_{i,y}}$, $\boldsymbol{E_{i,z}}$
5. Expect from symmetry: $\Sigma\,(\boldsymbol{E_{i,x}}+\boldsymbol{E_{i,y}}) = 0$
6. $\boldsymbol{E} = E_z\,\boldsymbol{e_z}$ ***only !***

1. Hollow sphere, homogeneously charged



Distributed charge: dQ

$$dE_z = \frac{dQ}{4\pi\varepsilon_0 r^2}(\boldsymbol{e_r} \bullet \boldsymbol{e_z})$$

charge element

$dQ = \sigma\, dA$

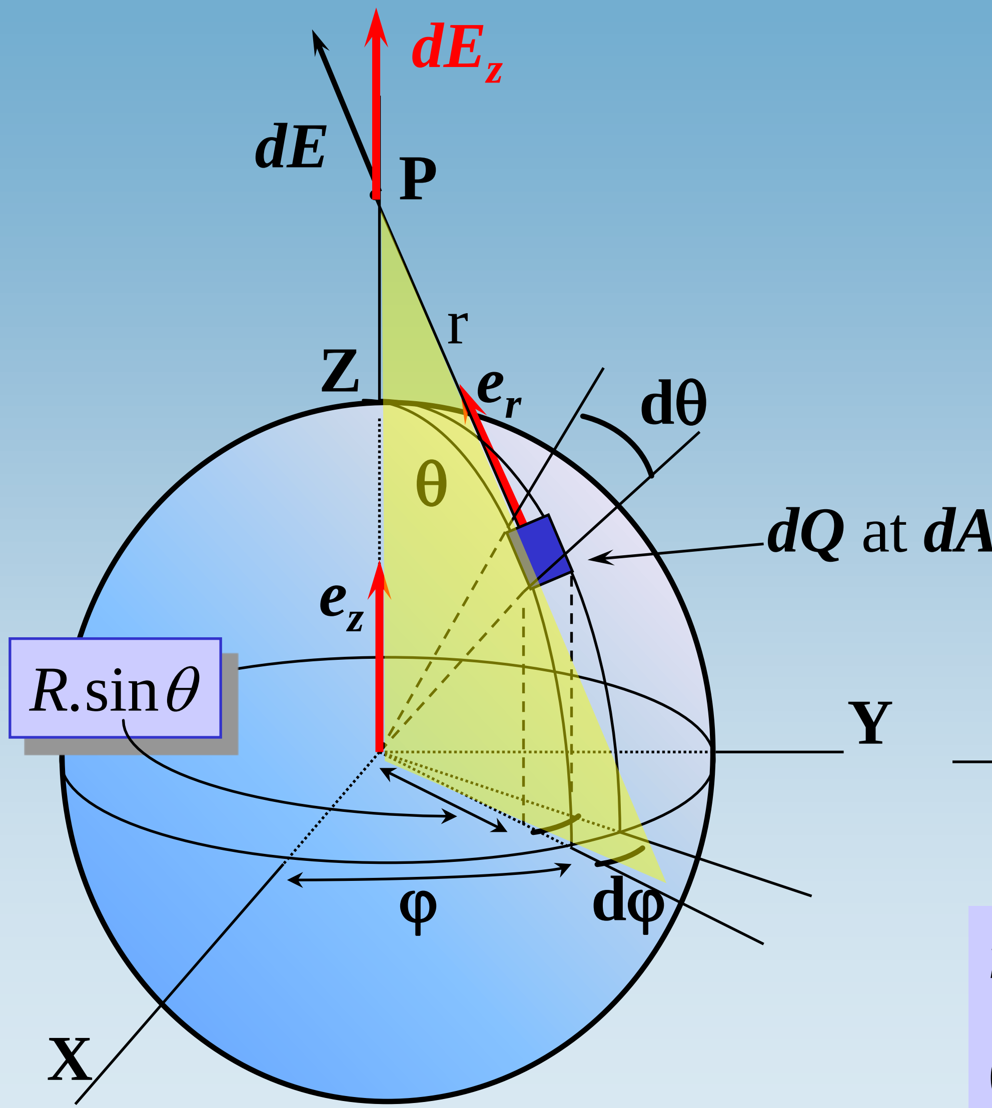
surface element

$dA=(R.d\theta).(R.\sin\theta.d\varphi)$

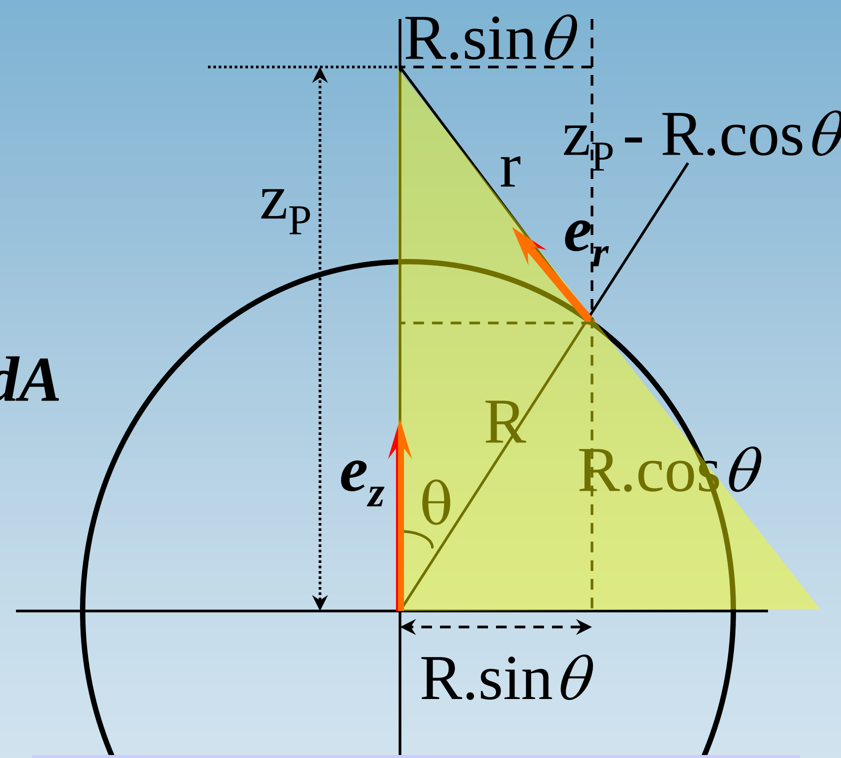
r and $(\boldsymbol{e_r} . \boldsymbol{e_z})$:

see next page

1. Hollow sphere, homogeneously charged



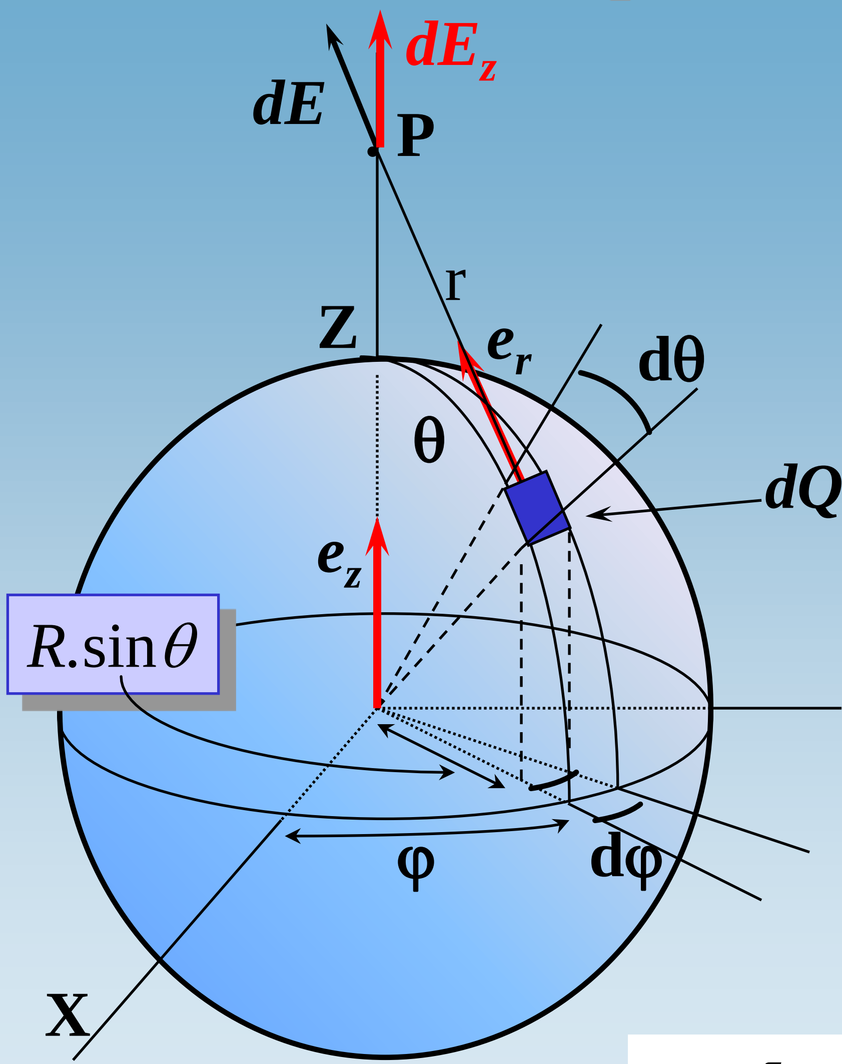
Cross section through OP:



$$r^2 = (R.\sin\theta)^2 + (z_P - R.\cos\theta)^2$$

$$(\boldsymbol{e}_r . \boldsymbol{e}_z) = (z_P - R.\cos\theta) / r$$

1. Hollow sphere, homogeneously charged



$$dE_z = \frac{\sigma\, dA}{4\pi\varepsilon_0 r^2}(\boldsymbol{e_r} \bullet \boldsymbol{e_z})$$

$dA=(R.d\theta).(R.\sin\theta.d\varphi)$

$r^2=(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2$

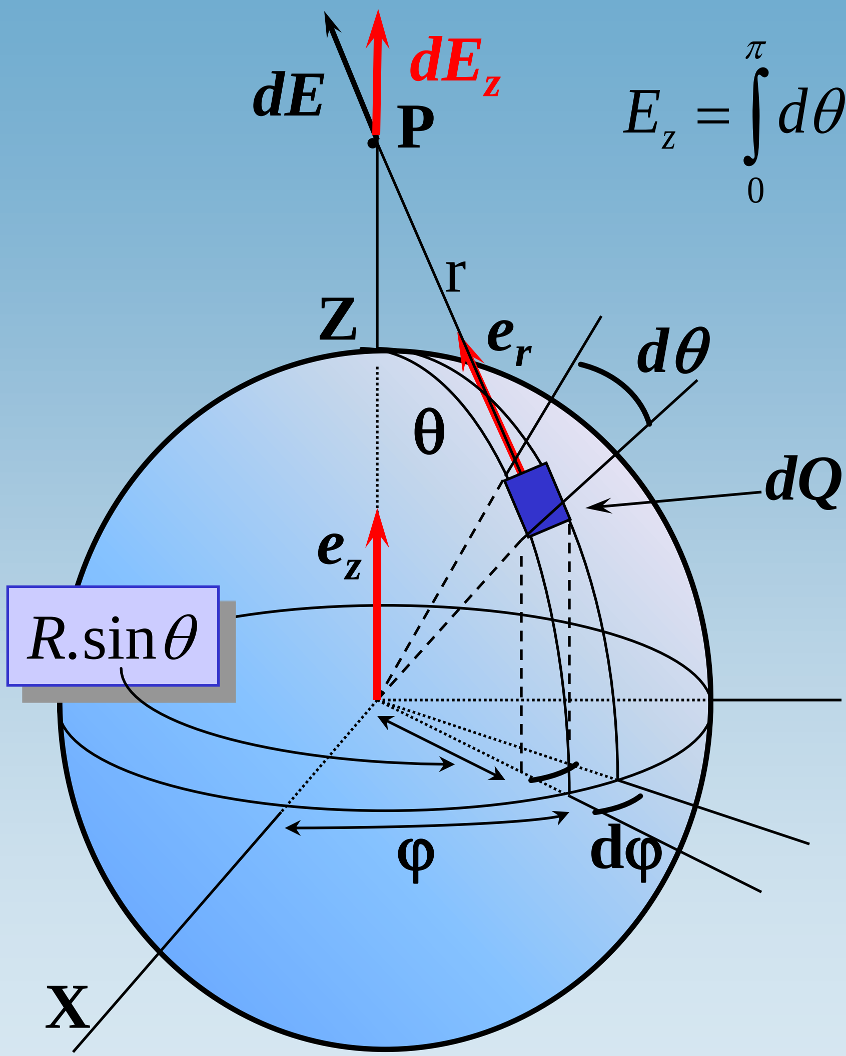
$(\boldsymbol{e_r} . \boldsymbol{e_z}) = (z_P - R.\cos\theta) / r$

These expressions can be used for both $|z_P| \geq R$ and $|z_P| < R$

$$dE_z = \frac{\sigma}{4\pi\varepsilon_0} \frac{R.d\theta.R.\sin\theta.d\varphi.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

$$E_z = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

1. Hollow sphere, homogeneously charged



$$E_z = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

Integration:

- over φ : results in factor 2π
- over θ : replace $\sin\theta.d\theta$ by $-d(\cos\theta)$,

  divide above and below by R^3,

  call $\cos\theta = x$, and $z_P/R = a$

  and $a^2+1-2ax = y$ (with $-2a.dx = dy$)

  and integrate over y

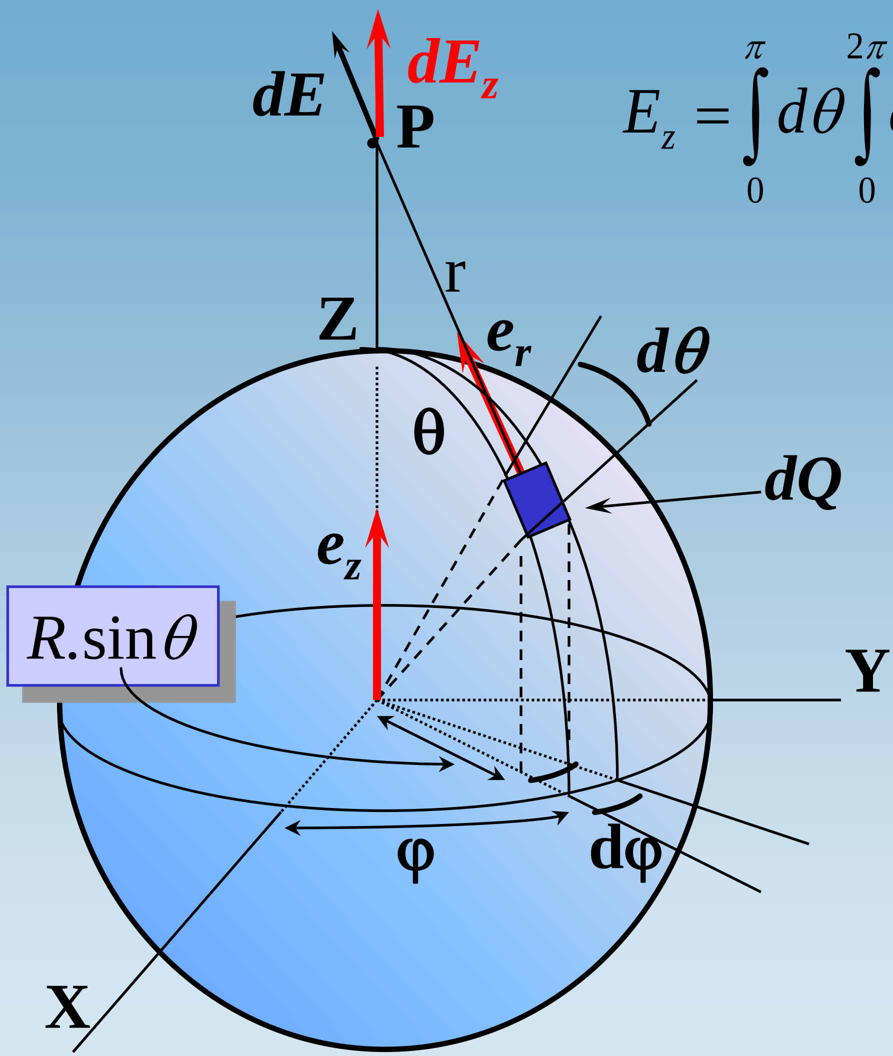
Result of integration:

$$E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \tfrac{1}{2}\left[\frac{-1}{\sqrt{y}} + \frac{1}{a^2\sqrt{y}} + \frac{\sqrt{y}}{a^2}\right]_{\theta=0}^{\theta=\pi}$$

2 cases: $|a| > 1$ and $|a| < 1$.

This expression can also be used for other limiting values of θ.

1. Hollow sphere, homogeneously charged



$$E_z = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

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with $a = z_P / R$ and $y = a^2 + 1 - 2a.\cos\theta$.

2 cases: $|a| > 1$ and $|a| < 1$.

This expression can also be used for other limiting values of θ.

Next slide: plot of F for varying values of the integration limits :
begin $\theta_b = 0..180^0$; end $\theta_e = 180^0$

$\theta_b = 0^0$: closed sphere
$\theta_b > 0^0$: “bowl” shape

1. Hollow sphere, homogeneously charged

$$E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \tfrac{1}{2}\left[\frac{-1}{\sqrt{y}} + \frac{1}{a^2\sqrt{y}} + \frac{\sqrt{y}}{a^2}\right]_{\theta=0}^{\theta=\pi}$$

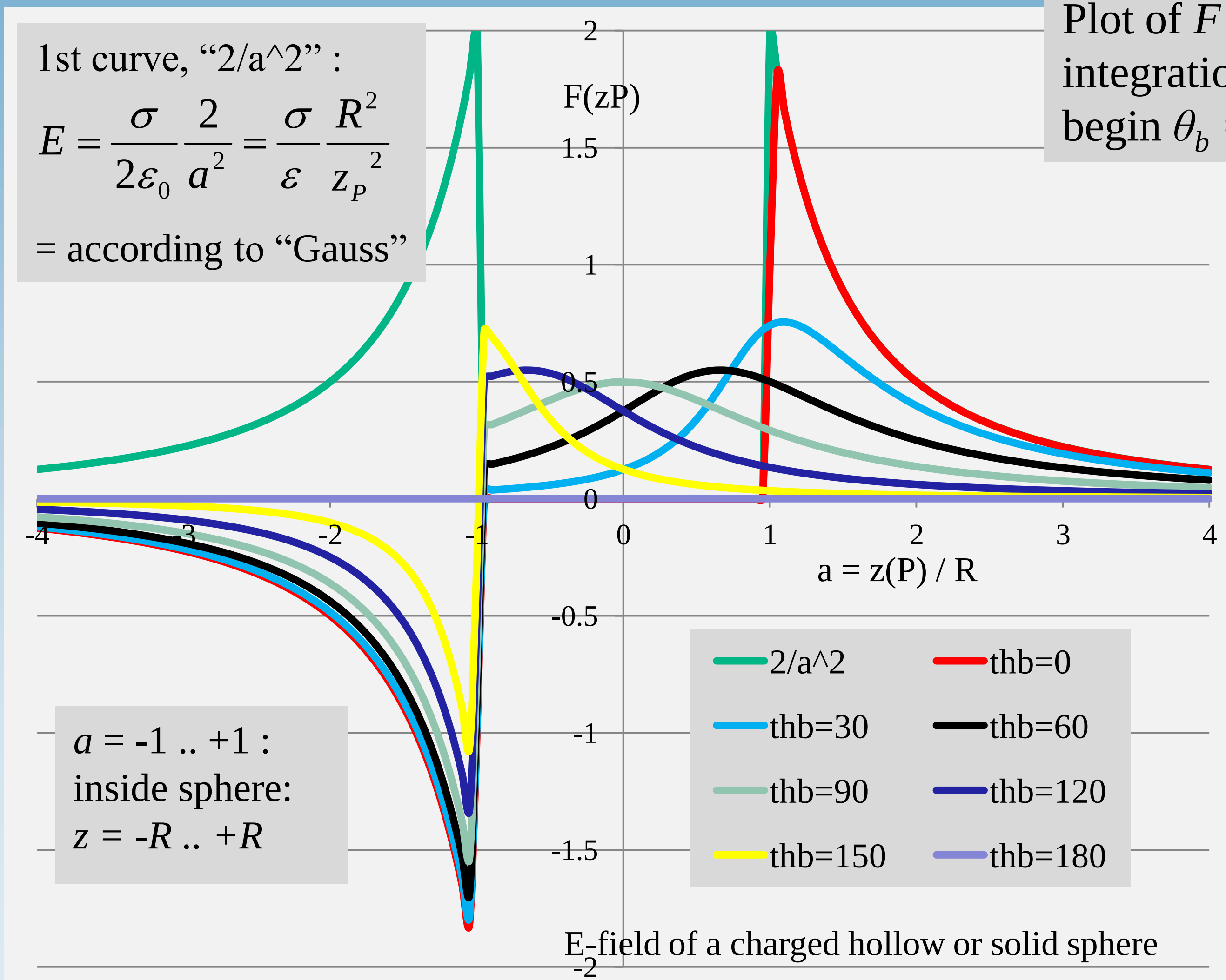
with $a = z_P/R$ and $y = a^2+1-2a.\cos\theta$.

Plot of F for varying values of the integration limits : begin θ_b = 0..180^0 ; end θ_e =180^0

1st curve, "2/a^2" :

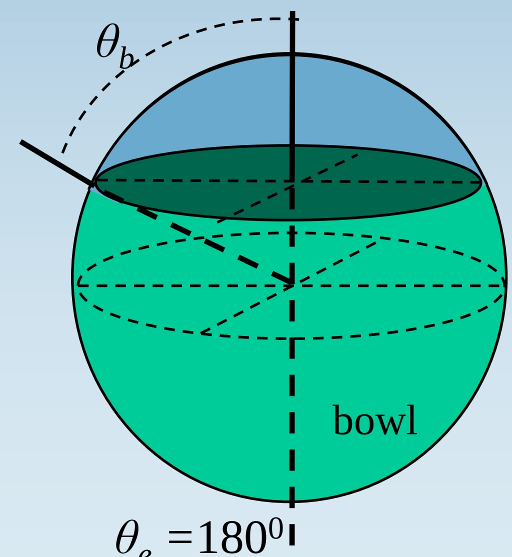
$$E = \frac{\sigma}{2\varepsilon_0}\frac{2}{a^2} = \frac{\sigma}{\varepsilon}\frac{R^2}{z_P{}^2}$$

= according to "Gauss"



a = -1 .. +1 : inside sphere: z = -R .. +R

$\theta_b = 0^0$: closed sphere
$\theta_b > 0^0$: "bowl" shape
In plot: θ_b = "thb"



1. Hollow sphere, homogeneously charged

$$E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \tfrac{1}{2}\left[\frac{-1}{\sqrt{y}} + \frac{1}{a^2\sqrt{y}} + \frac{\sqrt{y}}{a^2}\right]_{\theta=0}^{\theta=\pi}$$

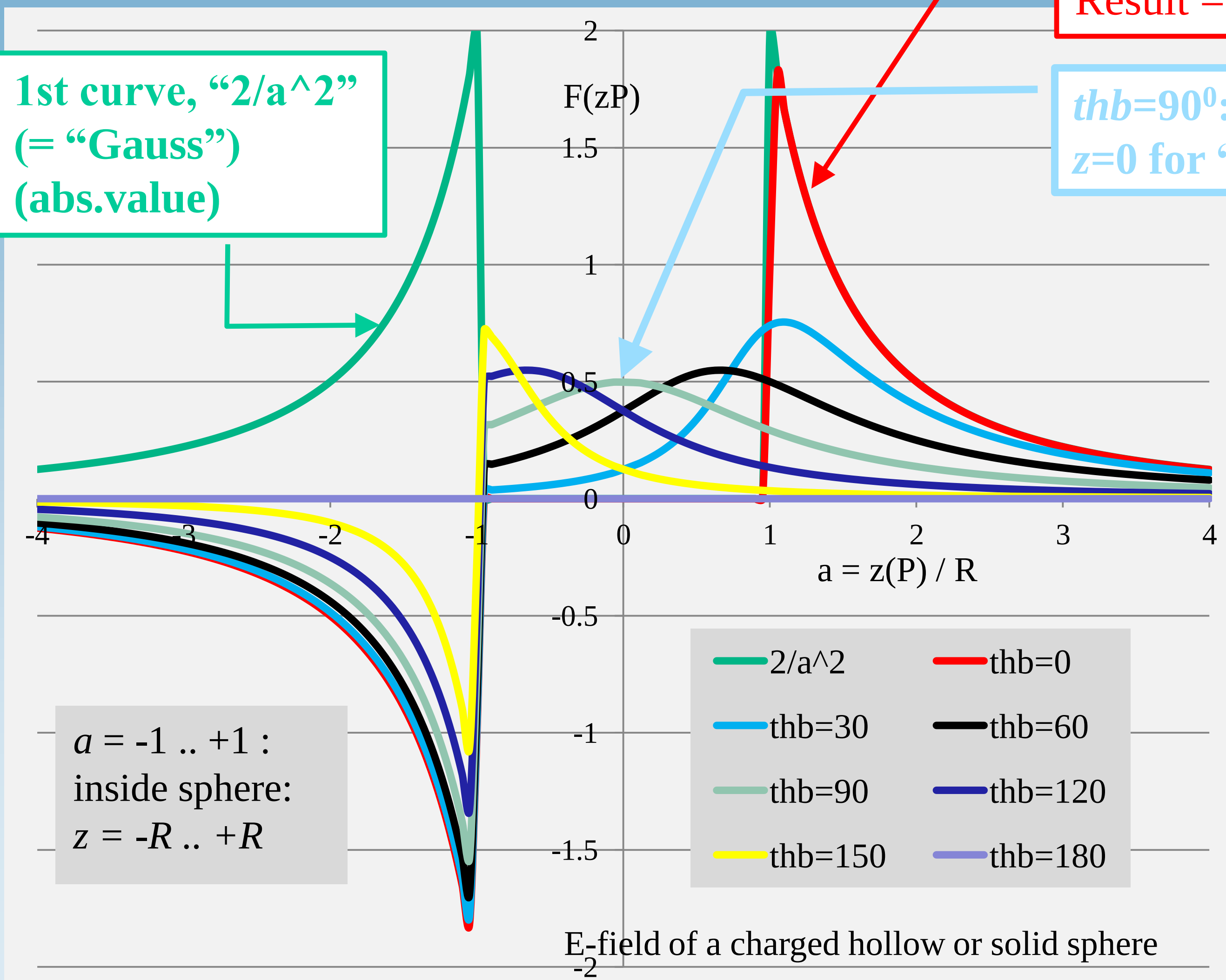
Correspondence: for *thb*=0 with **result from “Gauss”.**

Result = 0 inside sphere

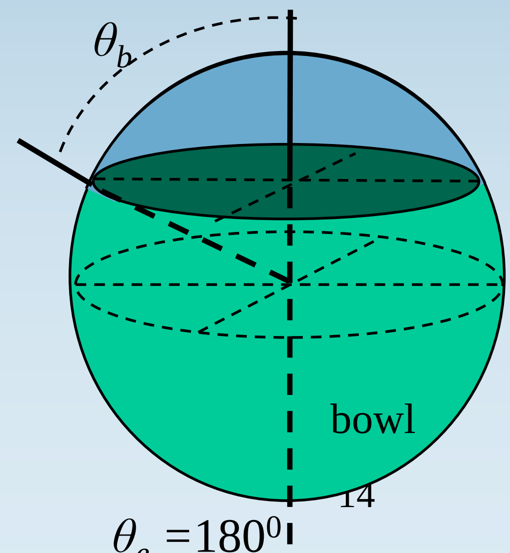
1st curve, “2/a^2” (= “Gauss”) (abs.value)

***thb*=90^0: Symmetric around *z*=0 for “half sphere”**

thb=30^0 and 150^0 :
thb=60^0 and 120^0 :
mutual symm. around *z*=0 inside sphere



a = -1 .. +1 :
inside sphere:
z = -*R* .. +*R*

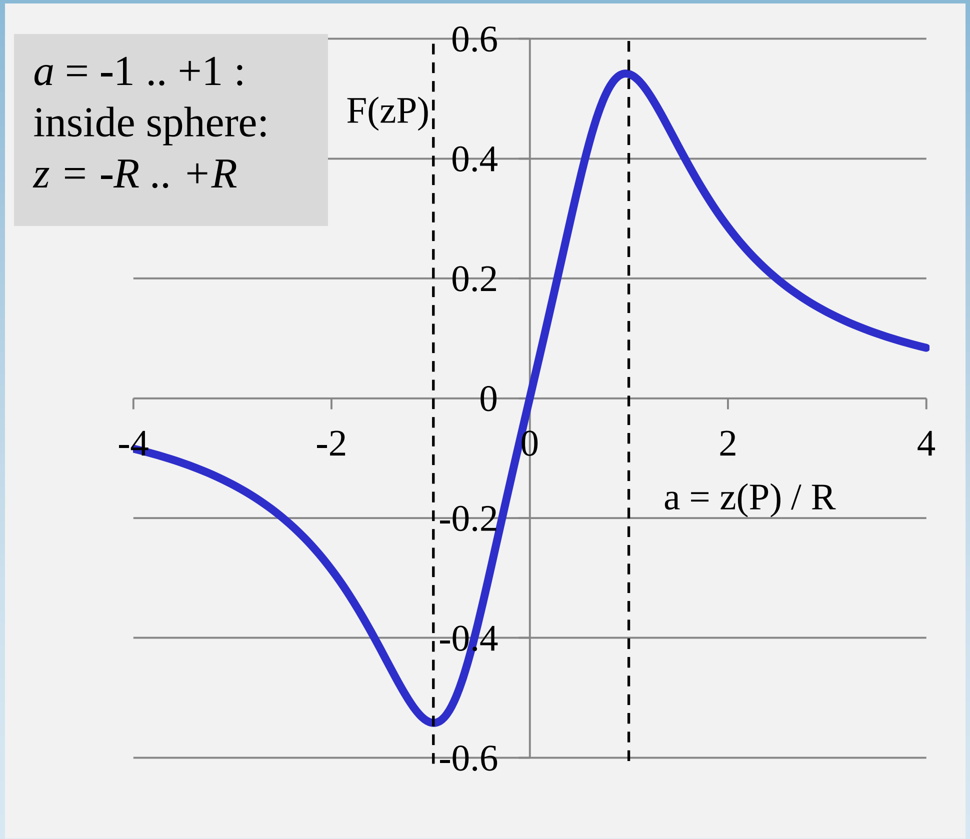


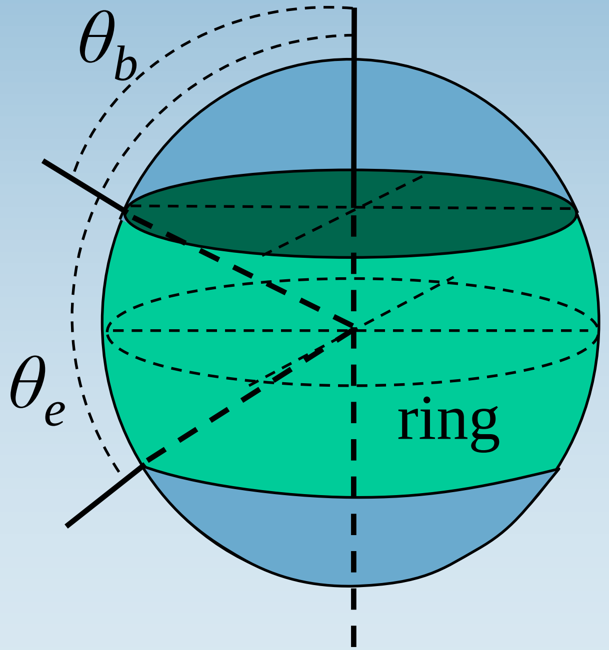
1. Hollow sphere, homogeneously charged

$$E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \tfrac{1}{2}\left[\frac{-1}{\sqrt{y}} + \frac{1}{a^2\sqrt{y}} + \frac{\sqrt{y}}{a^2}\right]_{\theta=0}^{\theta=\pi}$$

with $a = z_P/R$ and $y = a^2+1-2a.\cos\theta$.

Plot of F for integration limits : begin θ_b = 45; end θ_e =135^0 (ring)





1. Hollow sphere, homogeneously charged

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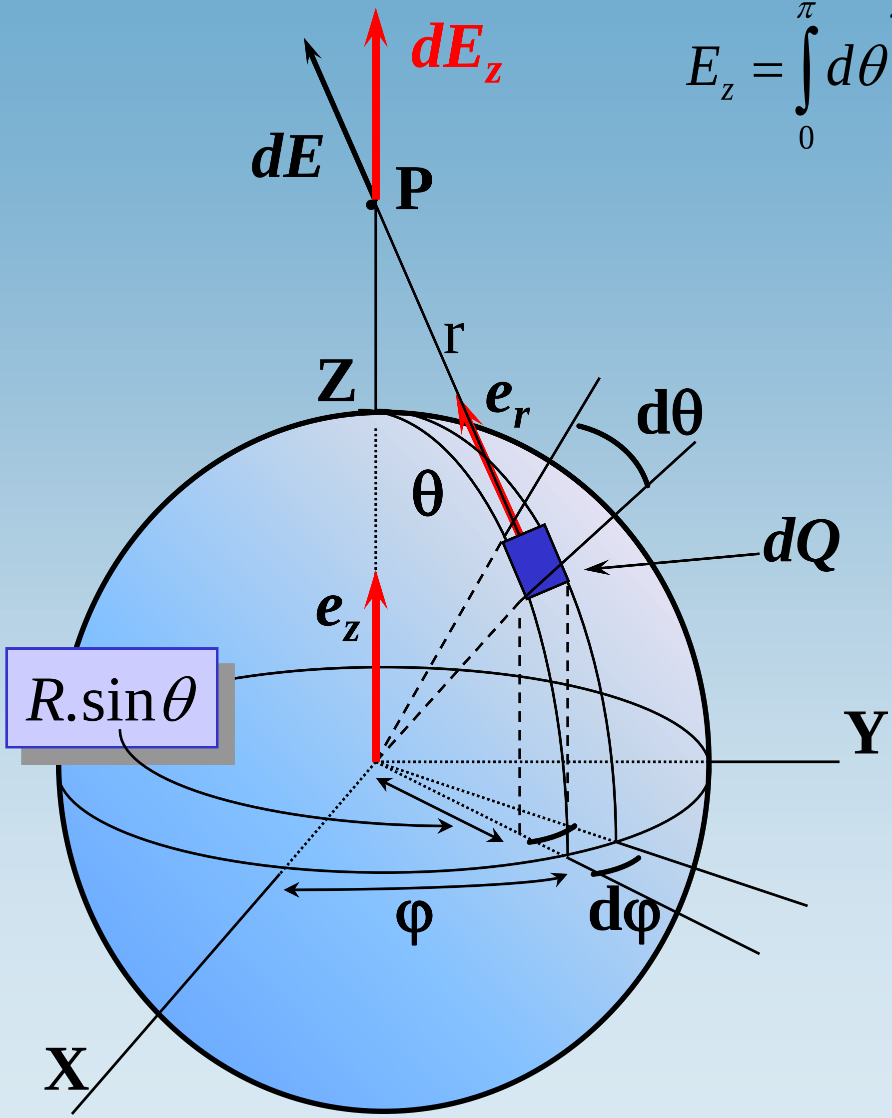

$$E_z = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

result for ***E*** in P:

$z_P < R$: ***E* = 0**

$z_P > R$: $\boldsymbol{E} = \frac{\sigma R^2}{\varepsilon_0 z_P^{\ 2}} \boldsymbol{e_z} = \frac{Q}{4\pi\varepsilon_0 z_P^{\ 2}} \boldsymbol{e_z}$

using $\sigma = Q/(4\pi R^2)$.

With Gauss:

Gauss sphere at radius z_P :

$$\oiint \boldsymbol{E}.\boldsymbol{dA} = \frac{Q_{encl}}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_0 z_P^{\ 2}}$$

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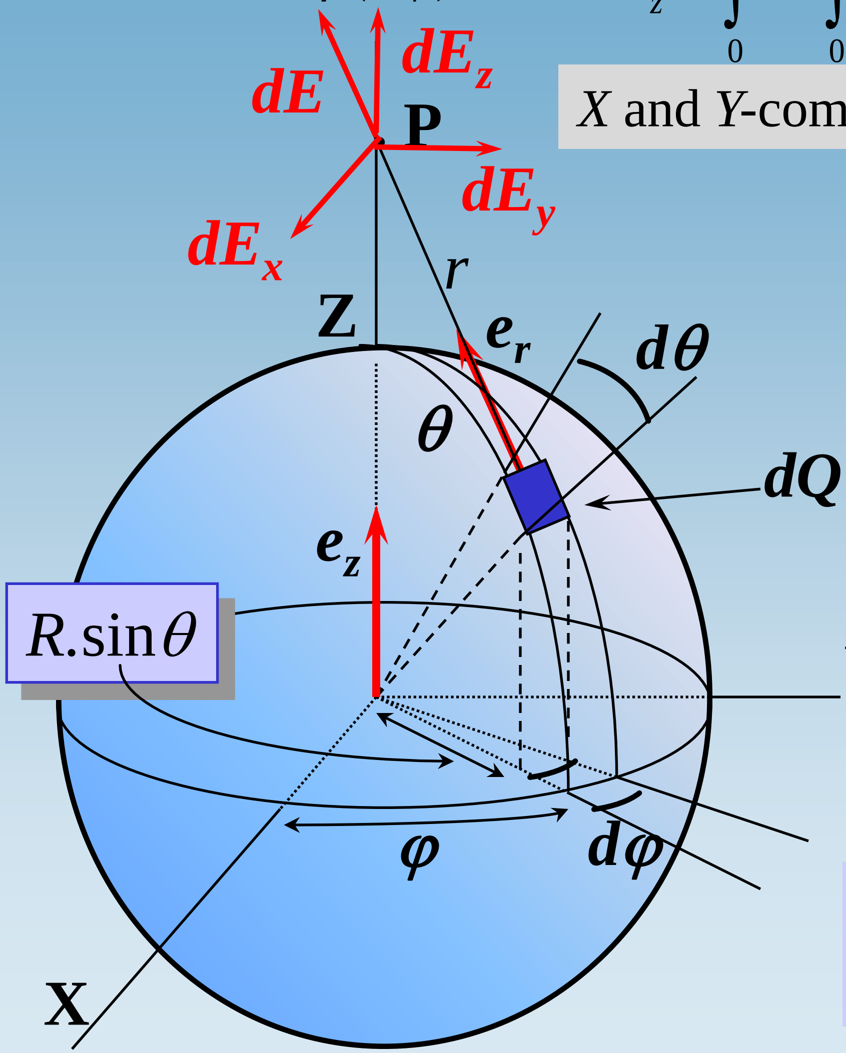
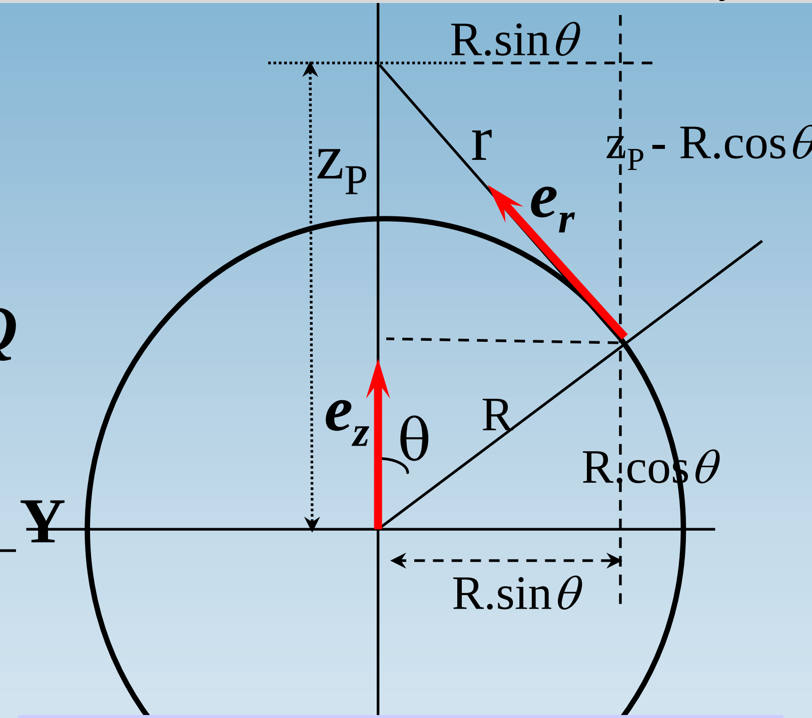
2. Hollow sphere, non-homogeneously charged

Now $\sigma = f(\theta,\varphi)$

$$E_z = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

X and Y-components may be present; $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are $\perp$ $\boldsymbol{e}_z$





$r^2 = (R.\sin\theta)^2 + (z_P - R.\cos\theta)^2$

For E_z : $(\boldsymbol{e}_r . \boldsymbol{e}_z) = (z_P - R.\cos\theta) / r$

For E_x and E_y we need: $R.\sin\theta \,/\, r$

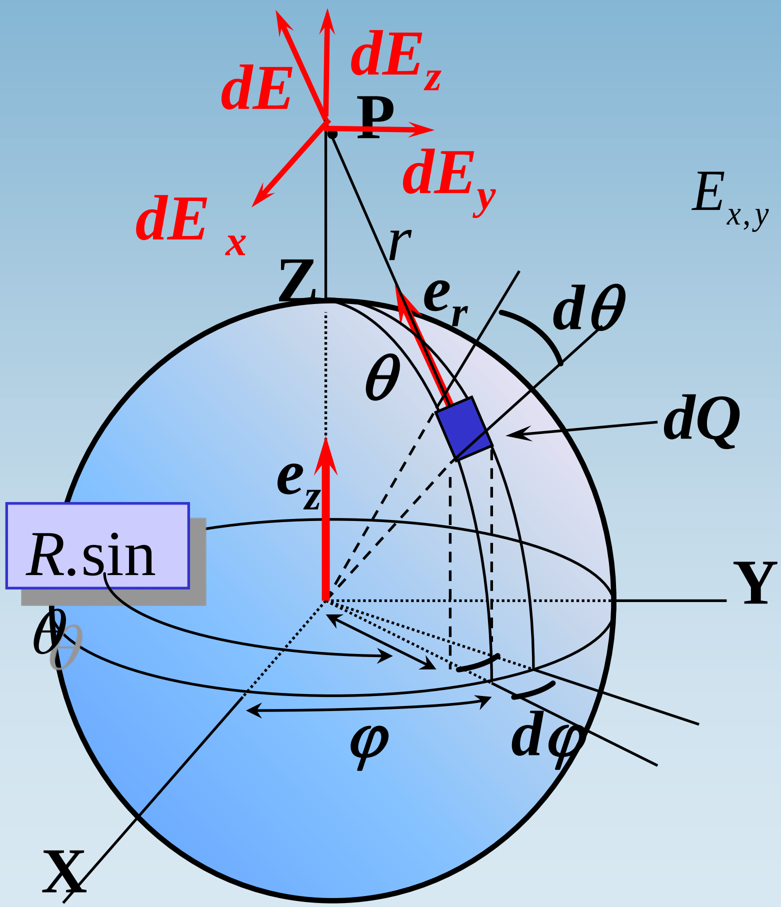
2. Hollow sphere, non-homogeneously charged

2. Hollow sphere, non-homogeneously charged

Now $\sigma = f(\theta, \varphi)$

$$E_z = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

For E_x and E_y we need: $R.\sin\theta / r$



$$E_{x,y} = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.\left[R.\sin\theta\right].\Phi(\varphi)}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

with $\Phi(\varphi) = \cos\varphi$ (for E_x);

$= \sin\varphi$ (for E_y).

In many situations, e.g. asymmetric charge distributions, numerical integration will be necessary.

E-field of a charged sphere

E-field of a charged sphere

Contents:

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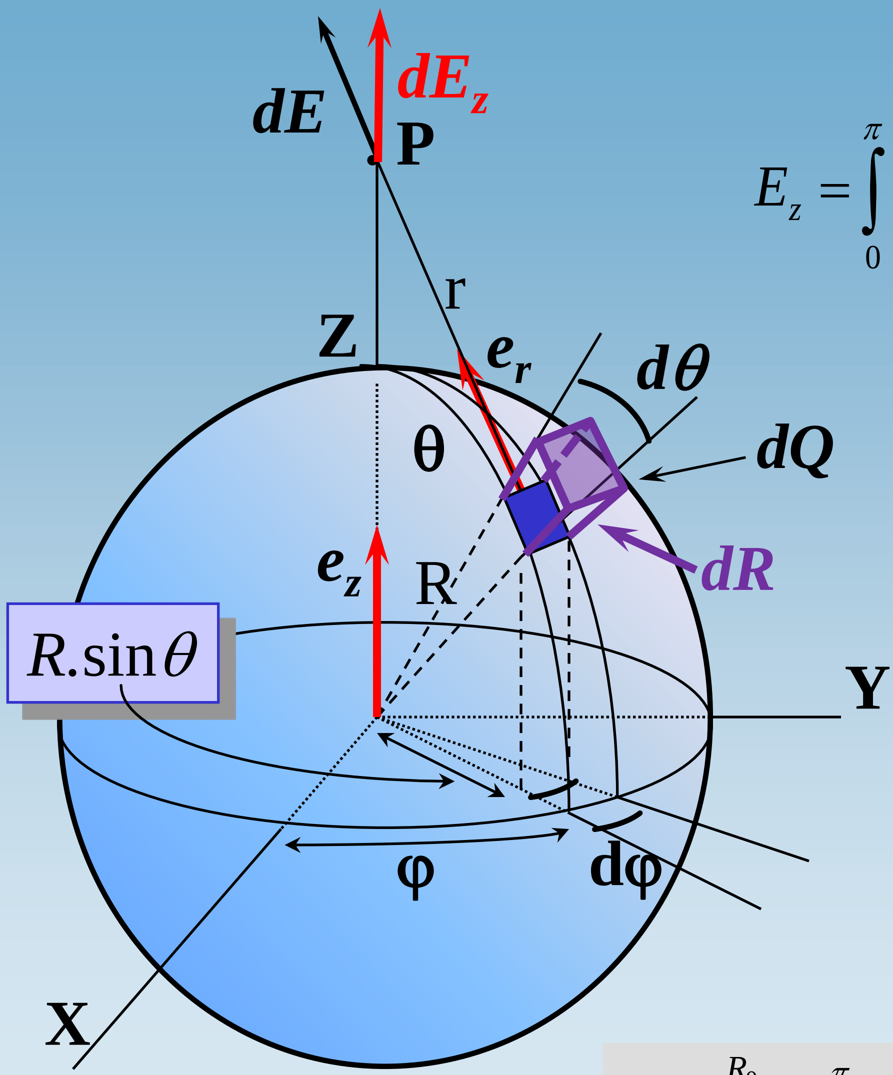
3. Solid sphere, homogeneously charged



Derived for a **hollow** sphere :

$$E_z = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

For a **solid** sphere we need an **extra integration variable:** over varying radius.

Suppose: radius R varies from 0 to R_0.

Integration element dR

Surface charge element

$dQ = \sigma.\ dA = \sigma\ .\ R.d\theta.R.\sin\theta.d\varphi$ (σ in C/m^2)

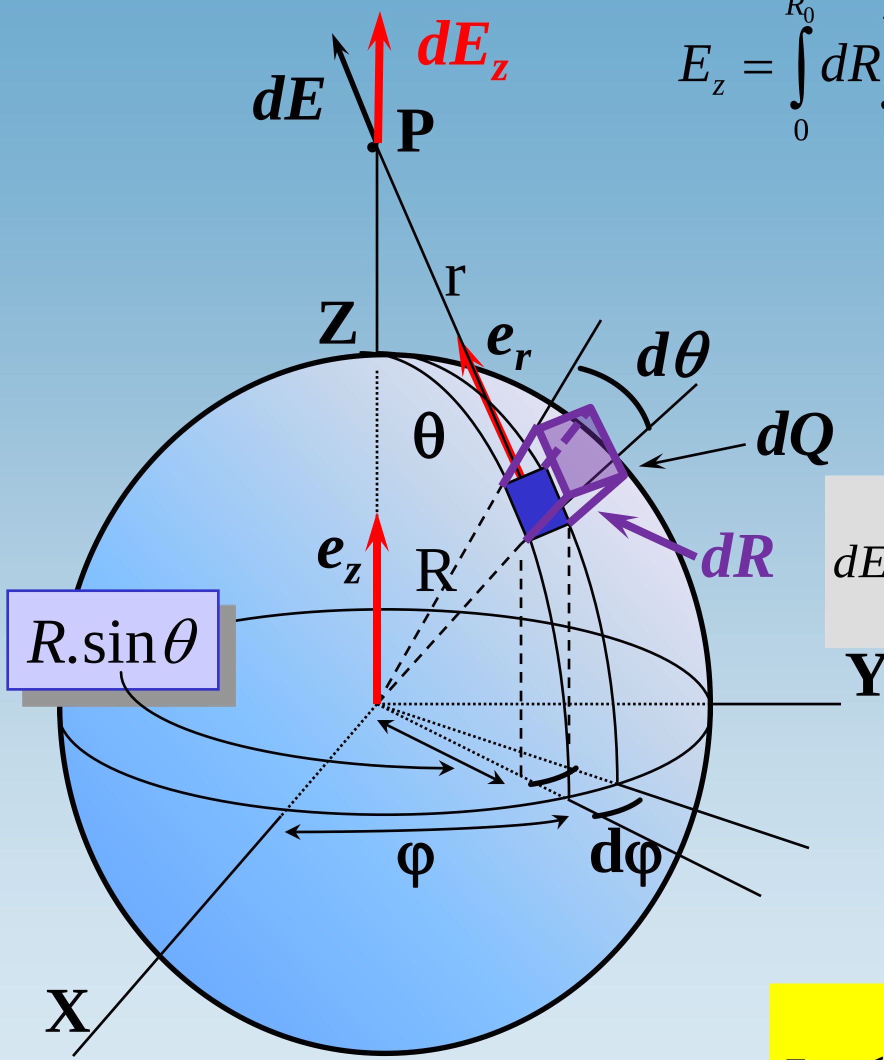
has to be replaced by

Volume charge element: (ρ in C/m^3)

$dQ = \rho.\ dV = \rho.\ dR.R.d\theta.R.\sin\theta.d\varphi$

$$E_z = \int_0^{R_0} dR \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\rho}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

3. Solid sphere, homogeneously charged



$$E_z = \int_0^{R_0} dR \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\rho}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

Integration over θ and φ resulted in:

$$E_z = \frac{\sigma F}{2\varepsilon_0}, \text{ with } F = \tfrac{1}{2}\left[\frac{-1}{\sqrt{y}} + \frac{1}{a^2\sqrt{y}} + \frac{\sqrt{y}}{a^2}\right]_{\theta=0}^{\theta=\pi}$$

with $a = z_P/R$ and $y = a^2 + 1 - 2a.\cos\theta$.

$$dE_z = \frac{\rho.dF}{2\varepsilon_0}, \text{ with d}F = \tfrac{1}{2}\left[\frac{-1}{\sqrt{y}} + \frac{1}{a^2\sqrt{y}} + \frac{\sqrt{y}}{a^2}\right]_{\theta=0}^{\theta=\pi} dR$$

Result for a hollow sphere:

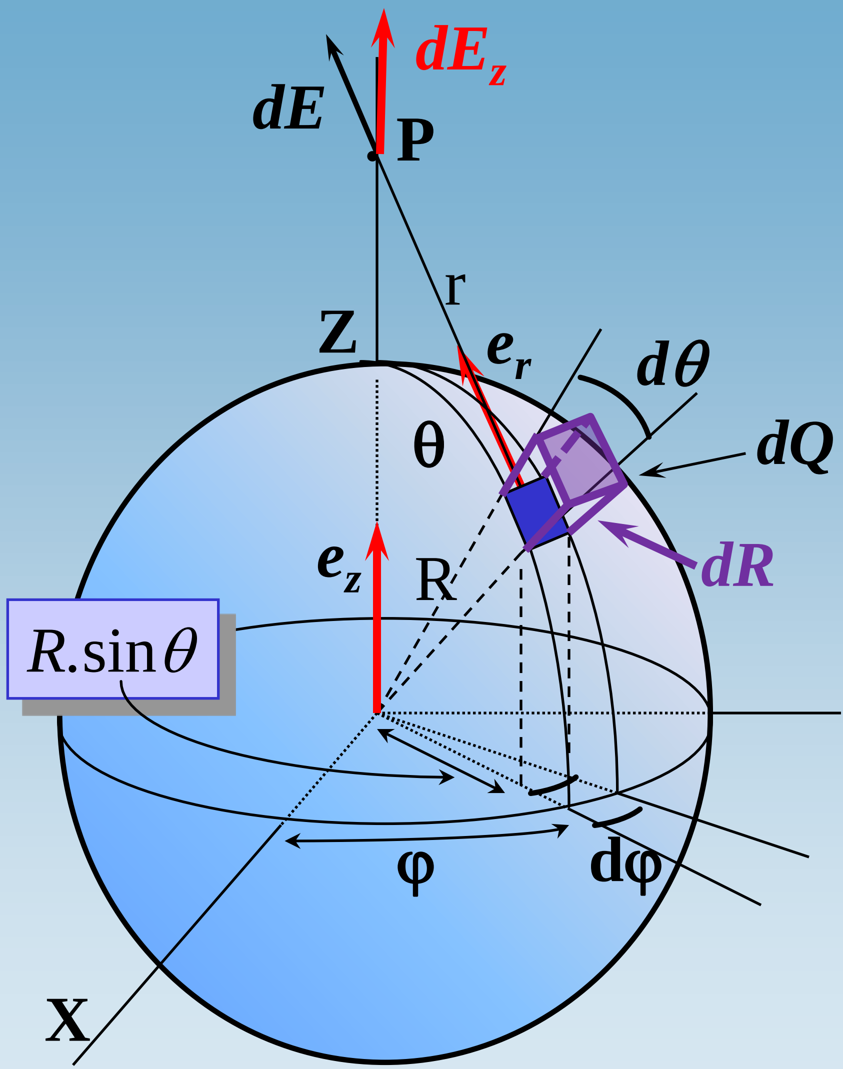
$$dE_z = \frac{\rho R^2}{\varepsilon_0 z_P^2} dR$$

$$z_P > R_0 : E_z = \int_0^{R_0} \frac{\rho R^2}{\varepsilon_0 z_P^2} dR = \frac{\rho R_0^3}{3\varepsilon_0 z_P^2}$$

$$z_P < R_0 : E_z = \int_0^{z_P} \frac{\rho R^2}{\varepsilon_0 z_P^2} dR = \frac{\rho z_P^3}{3\varepsilon_0 z_P^2} = \frac{\rho z_P}{3\varepsilon_0}$$

(sphere shells > z_P do not contribute !)

3. Solid sphere, homogeneously charged



Result for a solid sphere:

$$z_P > R_0 : E_z = \int_0^{R_0} \frac{\rho R^2}{\varepsilon_0 z_P{}^2} dR = \frac{\rho R^3}{3\varepsilon_0 z_P{}^2}$$

$$z_P < R_0 : E_z = \int_0^{z_P} \frac{\rho R^2}{\varepsilon_0 z_P{}^2} dR = \frac{\rho z_P{}^3}{3\varepsilon_0 z_P{}^2} = \frac{\rho z_P}{3\varepsilon_0}$$

And with $Q = \rho . (4/3) . \pi R_0{}^3$:

outside $: z_P > R_0 : E_z = \dfrac{Q}{4\pi\varepsilon_0 z_P{}^2}$ inverse quadratic

inside $: z_P < R_0 : E_z = \dfrac{Q . z_P}{4\pi\varepsilon_0 R^3}$ linear

These expression are according to “Gauss”.

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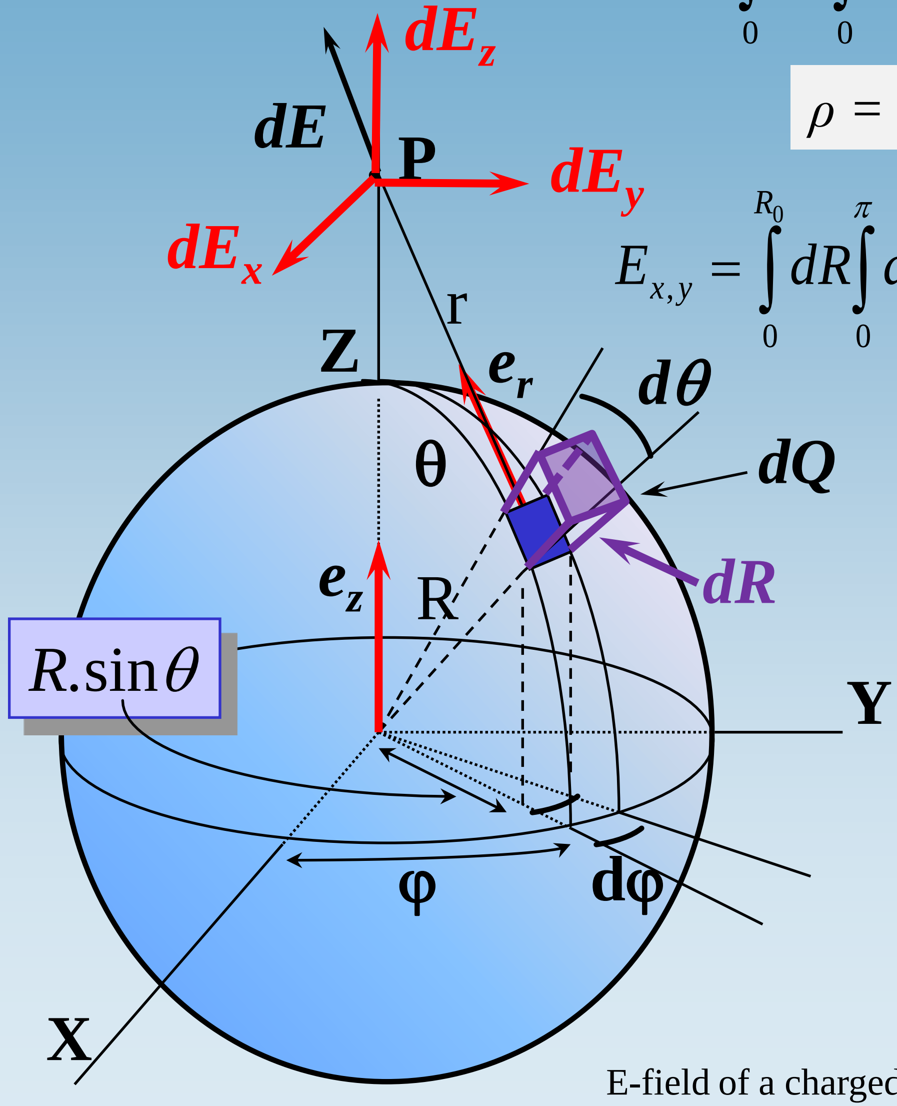
4. Solid sphere, non-homogeneously charged



$$E_z = \int_0^{R_0} dR \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\rho}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.[z_P - R.\cos\theta]}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

$\rho = f(R,\theta,\varphi)$: X- and Y-components present.

$$E_{x,y} = \int_0^{R_0} dR \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\rho}{4\pi\varepsilon_0} \frac{R^2.\sin\theta.\left[R.\sin\theta\right].\Phi(\varphi)}{[(R.\sin\theta)^2 + (z_P - R.\cos\theta)^2]^{3/2}}$$

with $\Phi(\varphi) = \cos\varphi$ (for E_x);

$= \sin\varphi$ (for E_y).

In general:
Integration has to be performed numerically.

Conclusion

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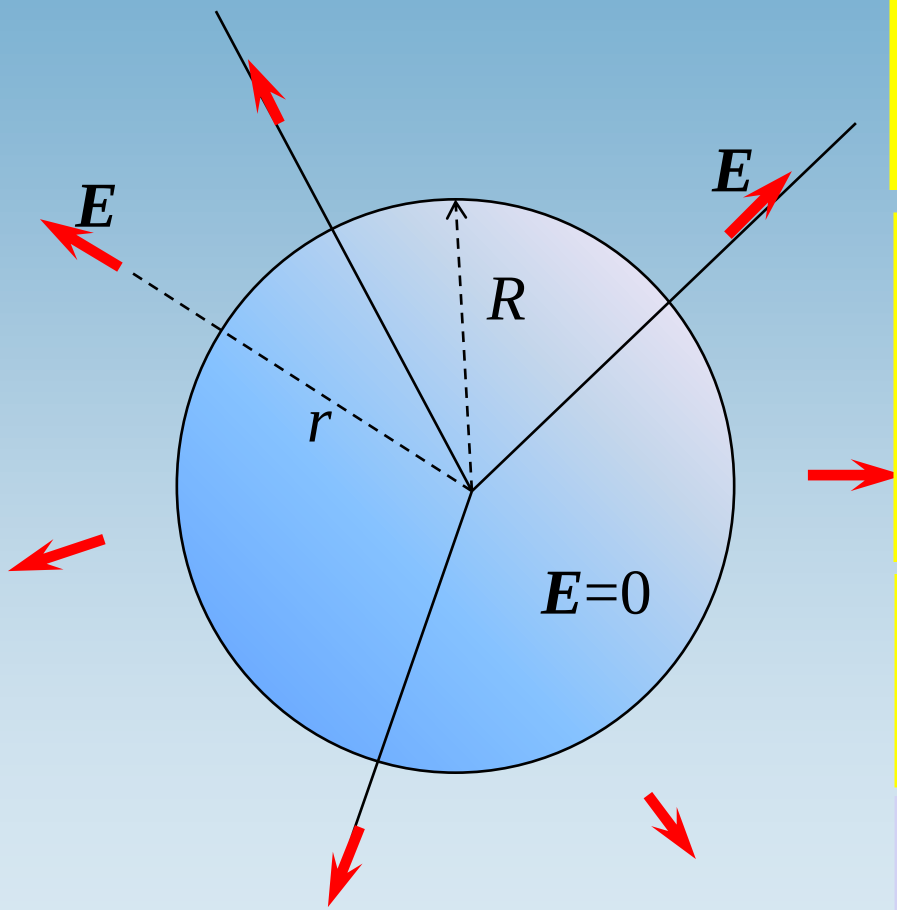

for homogeneous charge distribution:

$r > R$:	$\boldsymbol{E} = \dfrac{Q_{tot}}{4\pi\varepsilon_0 r^2}\boldsymbol{e_z}$
$r < R$: hollow:	$\boldsymbol{E} = 0$
solid:	$\boldsymbol{E} = \dfrac{Q_{tot}.r}{4\pi\varepsilon_0 R^2}\boldsymbol{e_z}$

Hollow: $Q_{tot} = \sigma.4\pi R^2$

Solid: $Q_{tot} = \rho.4\pi R^3/3$

total charge seems to be in center

the end