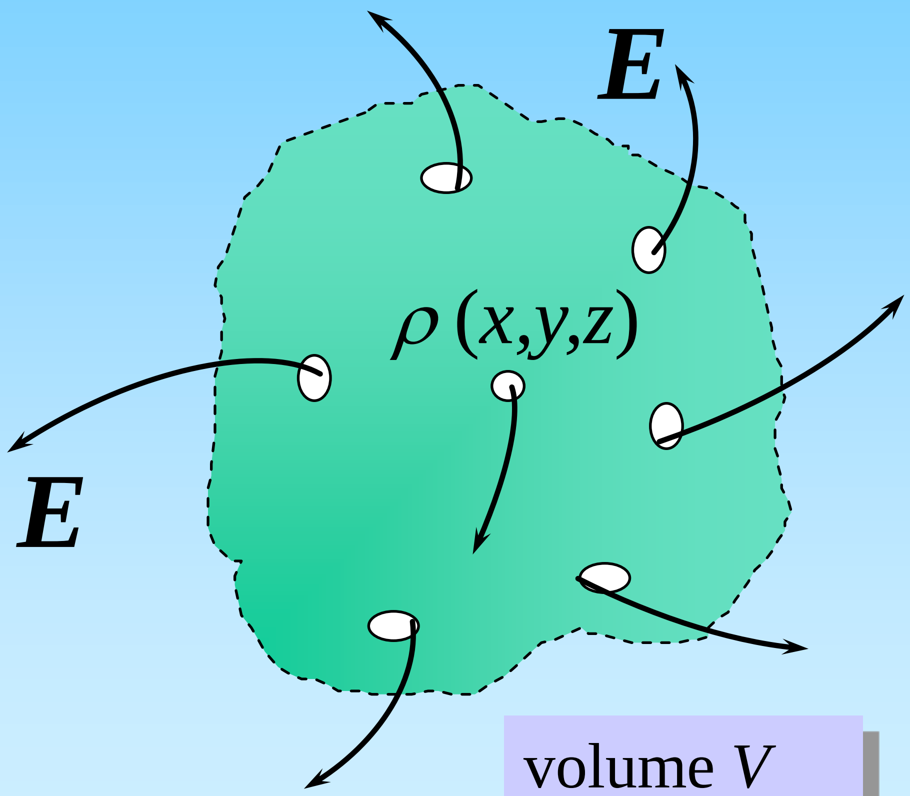
Energy in the Electric Field

Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

Energy in the Electric Field



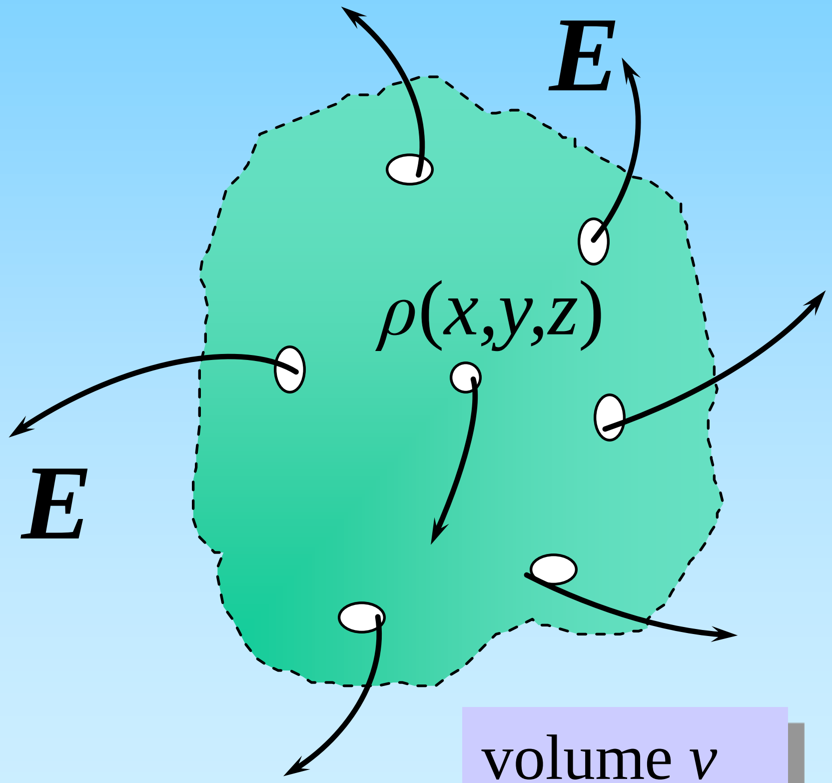
<u>Available:</u> Electric Field $\boldsymbol{E}$ produced by a distributed charge density ρ [C/m^3]

<u>Question:</u> How much energy did it cost to build up the electric field $\boldsymbol{E}$ by positioning the charge density ρ ???

volume V
surface A

Expressions for the Energy

Following expressions for the **field energy *W*** will be derived:



A. $W = \frac{1}{2}\iiint_V \rho_f . V \; dv$

ρ_f = density of free charges
V = potential function

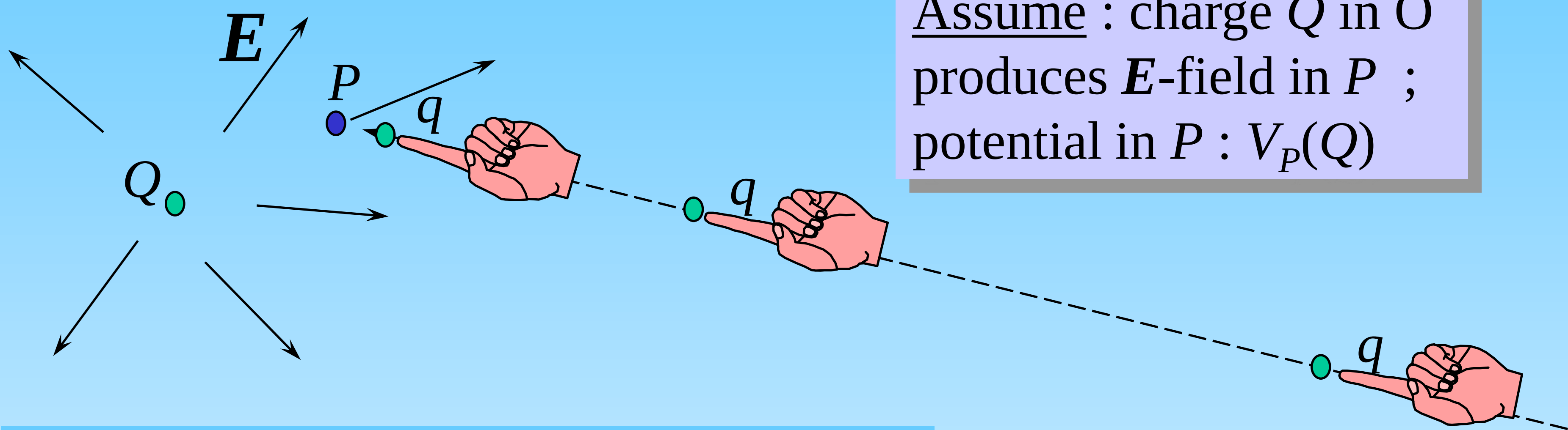
volume *v*
surface *A*

B. $W = \frac{1}{2}\iiint_V \mathbf{D} \bullet \mathbf{E} \; dv$

E = electric field

D = dielectric displacement

Energy = $f(\rho_f, V)$ (1)



<u>Assume</u> : charge Q in O produces $\boldsymbol{E}$-field in P ; potential in P : $V_P(Q)$

How much energy W is needed to bring charge q from infinity to P ?

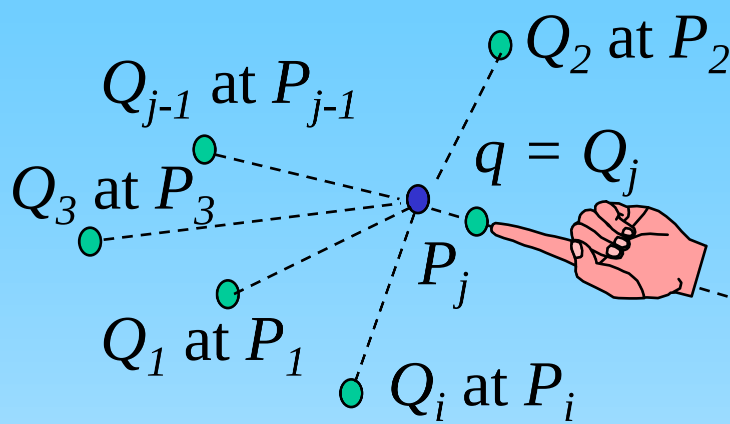
$$W = q.V_P(Q)$$

Suppose N charges Q_i (i=1..N) in O :

Each charge produces its own V in P

$$W = q.V_P \; ; \; V_P = \sum_{i=1..N} V_P(Q_i)$$

Energy = $f(\rho_f, V)$ (2)



Suppose *j-1* charges Q_i (i=1..j-1), not necessarily all at O , but at P_i

Call $q=Q_j$, to be placed at $P=P_j$

Energy needed :

- to place *1st* charge Q_1 at P_1 :
- to place *2nd* charge Q_2 at P_2 :
- to place *3rd* charge Q_3 at P_3 :

$$W_1 = 0$$

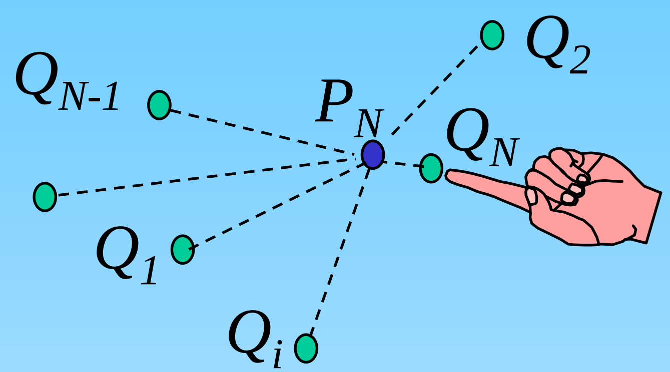
$$W_2 = Q_2 . V(Q_1)$$

$$W_3 = Q_3 . \left[V(Q_1) + V(Q_2) \right]$$

Total energy needed to position all *N* charges Q_j at P_j (j=1..N) , with preceding Q_i (i=1..j-1) present :

$$W = \sum_{j=1..N} Q_j \sum_{i=1..j-1} V_{P_j}(Q_i)$$

Energy = $f(\rho_f, V)$ (3)



Total energy needed to position all N charges Q_j at P_j (j=1..N) , with preceding Q_i (i=1..j-1) present :

$$W = \sum_{j=1..N} Q_j \sum_{i=1..j-1} V_{P_j}(Q_i) \quad = \frac{1}{2} \sum_{j=1..N} Q_j \sum_{i=1..N;\, i\neq j} V_{P_j}(Q_i)$$

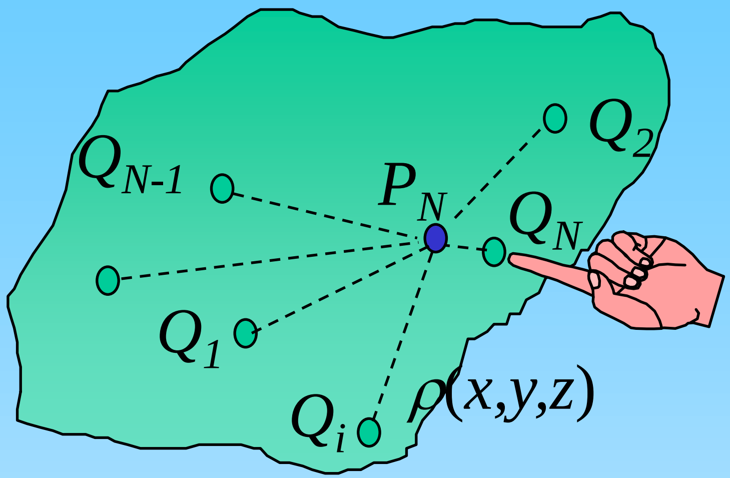
Summation i,j=1..N ; factor 1/2 to avoid “double-count”

(note the summation limits)

Summation is over all charges, each in field of all other charges.

Energy = $f(\rho_f, V)$ (4)

$$W = \frac{1}{2} \sum_{j=1..N} Q_j \sum_{i=1..N;\, i \neq j} V_{P_j}(Q_i)$$



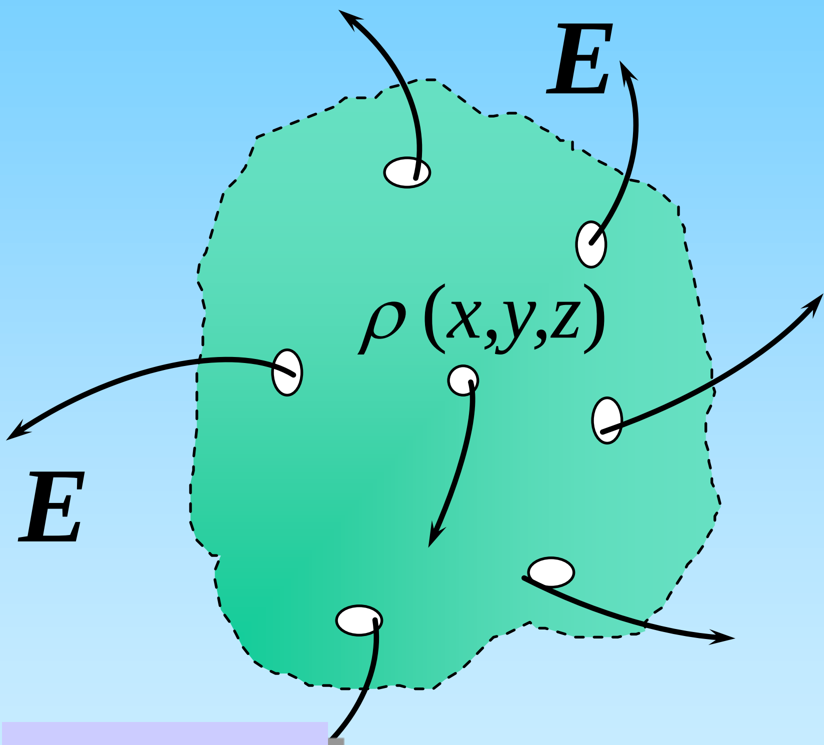
Suppose all charges are distributed as charge density ρ [C/m^3] :

"Summation over all charges, each in field of all other charges" now means:

1. Divide *v* into <u>volume elements</u> *dv* , with charge $\rho.dv$
2. Calculate <u>potential</u> from all other charges at that spot.
3. <u>Integrate</u> over volume *v* :

$$W = \frac{1}{2} \iiint_V \rho . V \, dv$$

Energy = $f(\rho_f, V)$ (5)



volume v
surface A

$$W = \frac{1}{2}\iiint_V \rho . V \; dv$$

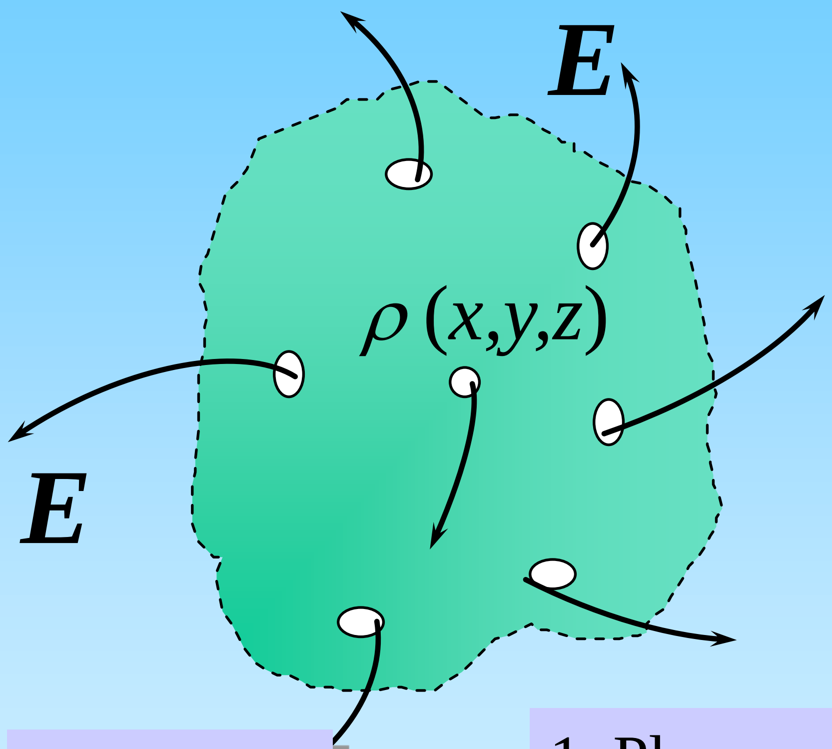
If dielectric material present:

V originates from all charges (free and bound) ;

ρ originates from free charges only (being "transportable") :

$$W = \frac{1}{2}\iiint_V \rho_f . V \; dv$$

Energy = $f(\boldsymbol{D}, \boldsymbol{E})$ (1)



Energy to build charge distribution = energy to destroy it.

How ?

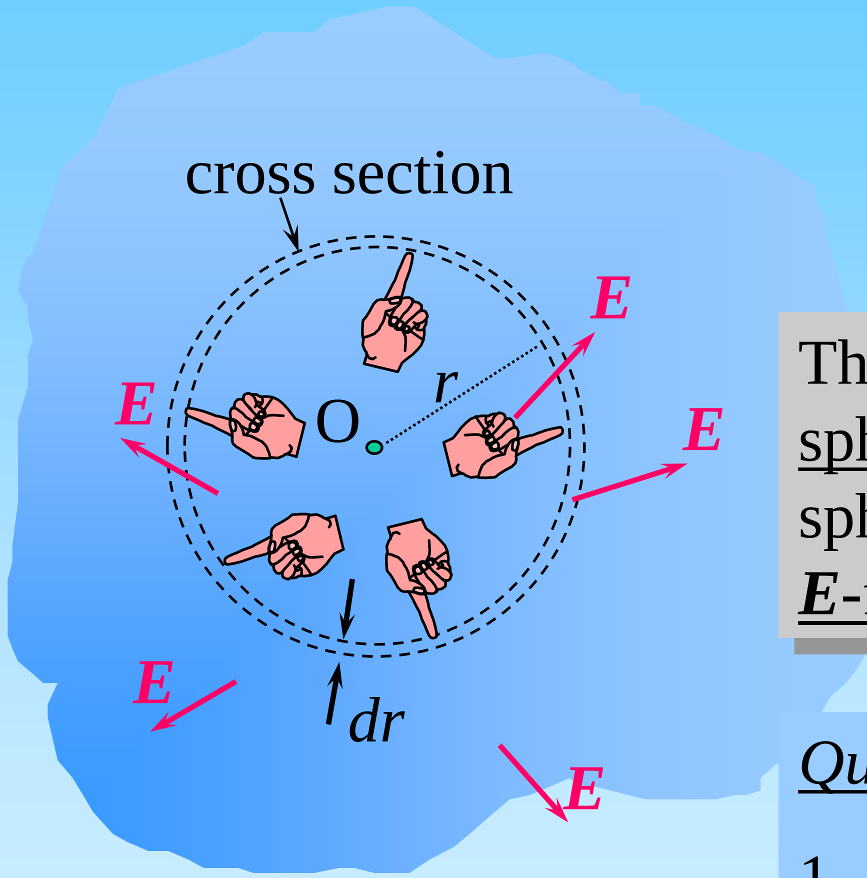
Transport all charges to infinity and record energy.

How ?

volume v
surface A

1. Place conducting sphere with radius = 0 at O.
2. "Blow up" till radius = infinity.
3. During blow-up: "Sweep" all charges to ∞.

Energy = f (*D*,*E*) (2)



Suppose: now radius = r

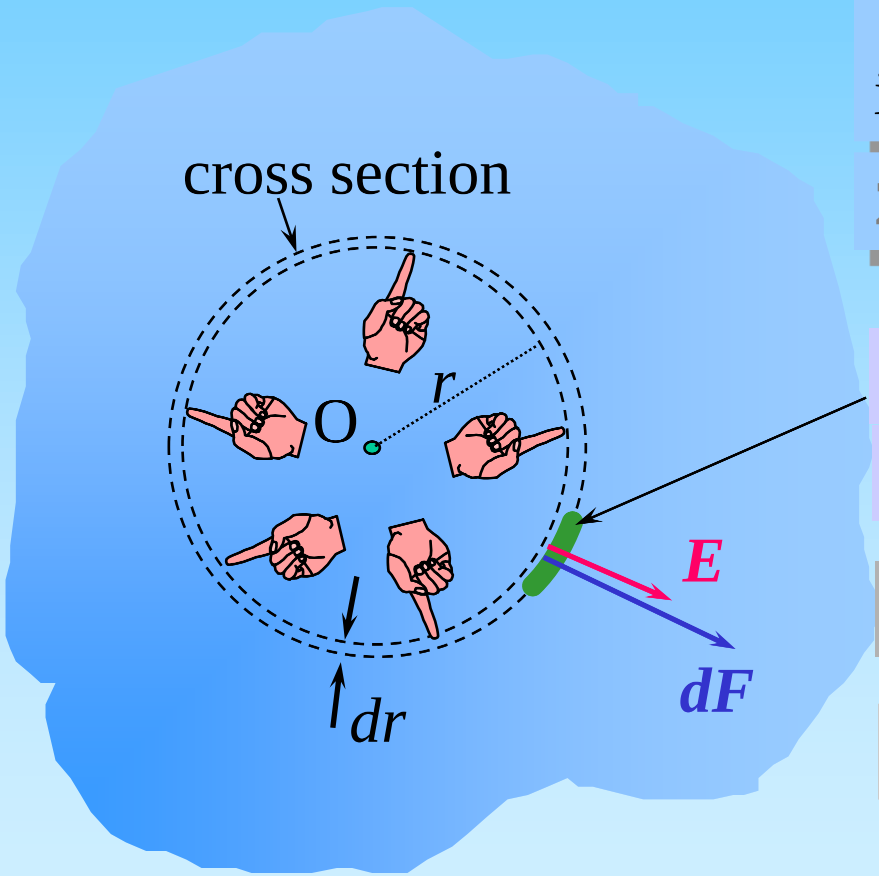
"Blowing up" the sphere:

Those charges originally <u>inside the sphere</u>, now lie <u>on the surface</u> of the sphere, and produce <u>same (average) ***E*-field**</u> as if they were inside (Gauss).

Questions:

1. how much energy is involved in increasing radius from r to $r+dr$??
2. integrate answer from $r=0$ to ∞.

Energy = $f\,(\mathbf{D},\mathbf{E})$ (3)



Questions:

1. How much energy is involved in increasing radius from r to $r+dr$??
2. integrate answer from $r=0$ to ∞.

Consider surface element dA

Free charge in dA: $dQ_f = \sigma_f\,.dA$

Work to shift dA from r to $r+dr$:

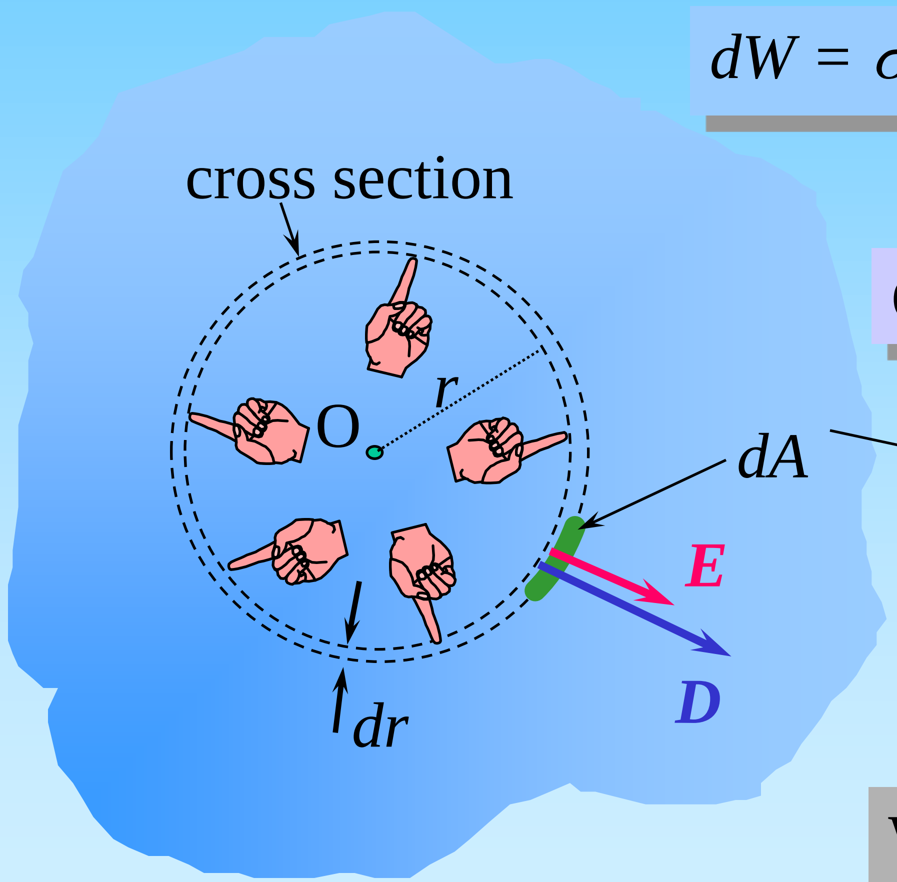
$dW = \mathbf{dF.dr} = dQ_f\,.\mathbf{E.dr}$

$dW = \sigma_f\,.dA.\mathbf{E.dr} = \sigma_f\,.\,E\,dv$

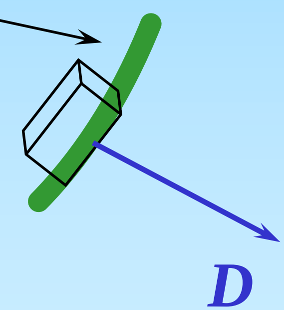
Energy = f ($\boldsymbol{D},\boldsymbol{E}$) (4)

Work needed to shift dA from r to $r+dr$:

$dW = \sigma_f \,.dA.\boldsymbol{E}.\boldsymbol{dr} \quad = E.\sigma_f \,.\, dv$



Question: what are $\boldsymbol{E}$ and σ_f ??



Gauss pill box:

$\sigma_f \,.dA = \boldsymbol{D}.\boldsymbol{dA}$

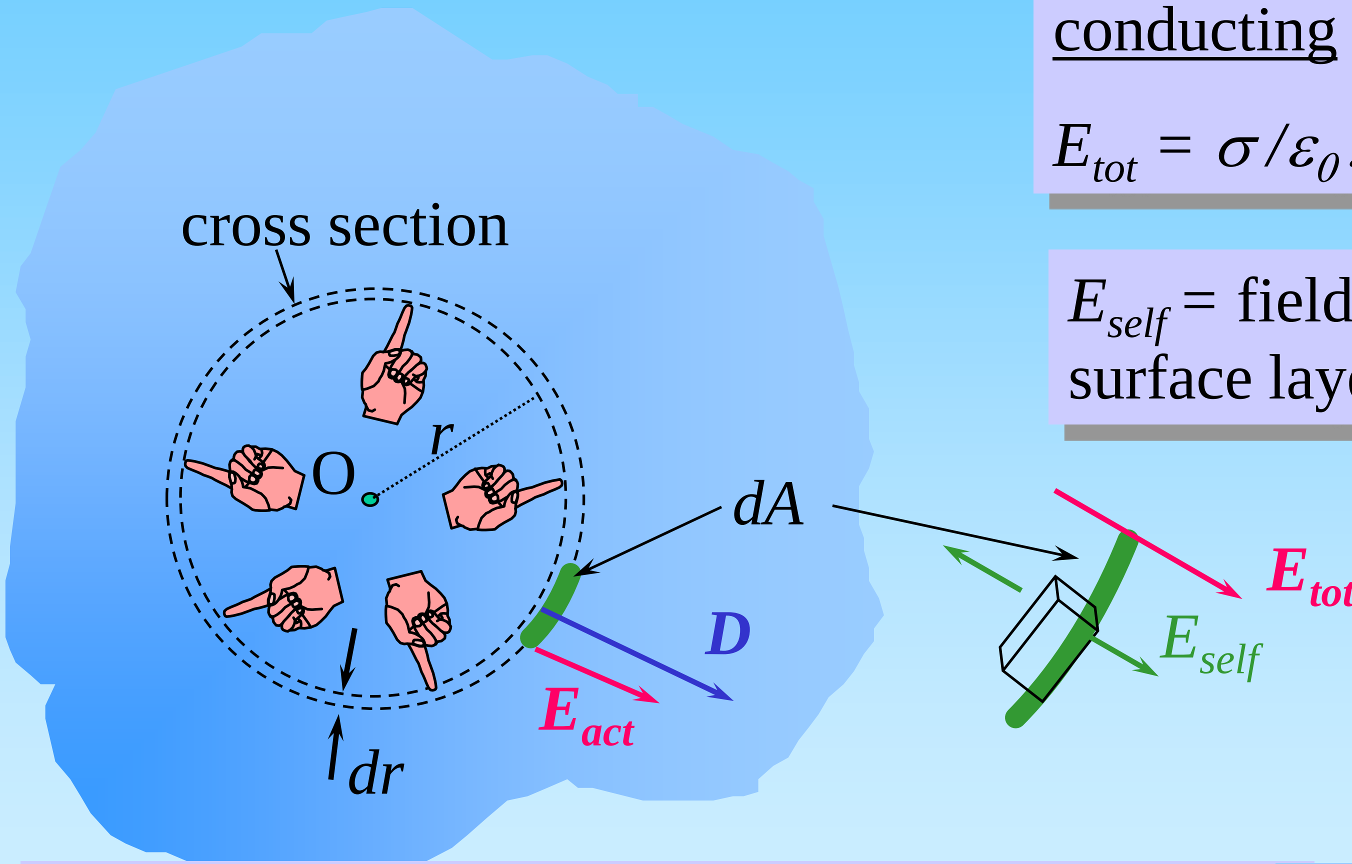
What $\boldsymbol{E}$ acts on dA? see next slide

Energy = $f\,(\boldsymbol{D},\boldsymbol{E})$ (5)

E_{tot} = total field, just outside conducting sphere:

$E_{tot} = \sigma/\varepsilon_0$.

E_{self} = field produced by surface layer itself:

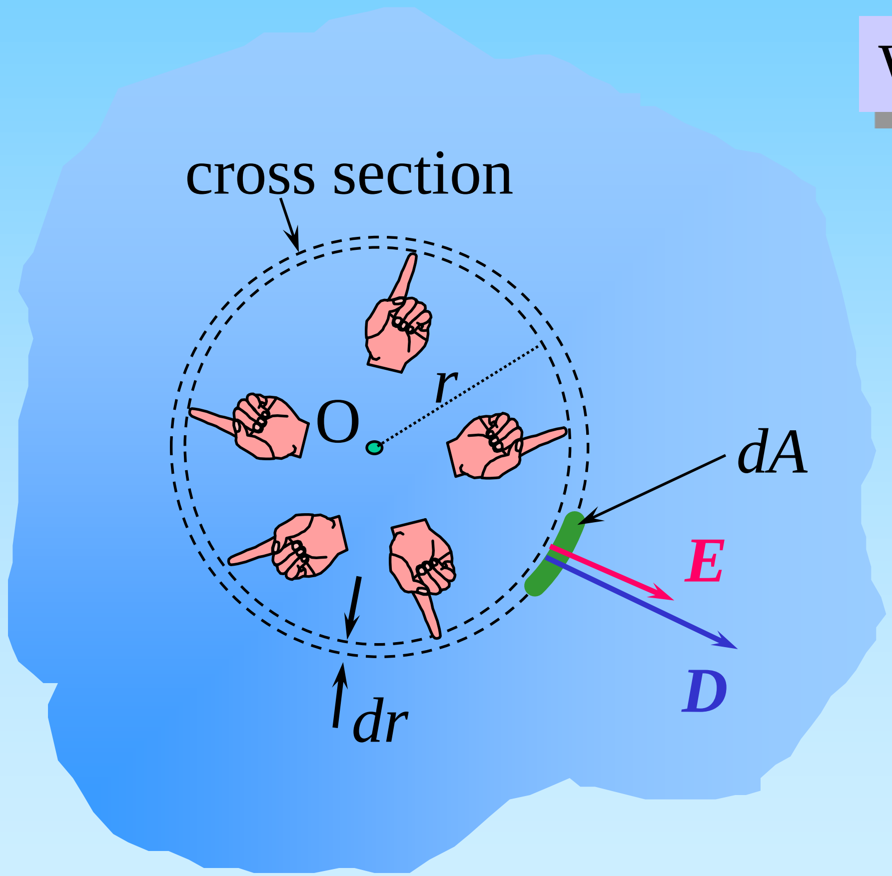


$E_{self} = {}^1/_2\, \sigma/\varepsilon_0$

Electric field **acting** on charge in pill box :
(apparently due to all "other" charges)

$E_{act} = E_{tot} - E_{self}$
$= {}^1/_2\, E_{tot}$

Energy = $f\,(\mathbf{D},\mathbf{E})$ (6)



Work to replace dA from r to $r+dr$:

$dW = \sigma_f\,.dA.\mathbf{E.dr} \;= E.\sigma_f\,.\,dv$

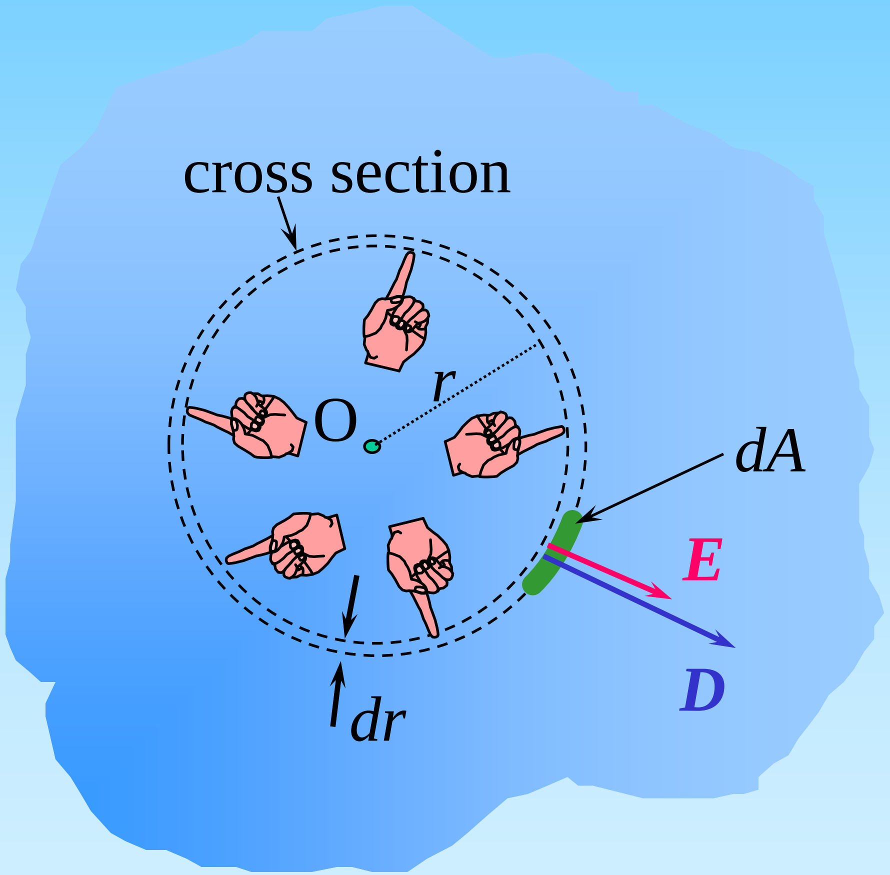
Gauss pill box: $\sigma_f\,.dA = \mathbf{D.dA}$

Acting $E_{act} = {}^1/_2\,E_{tot}\;(= {}^1/_2\,E\,)$

$dW = {}^1/_2\,\mathbf{D.E.dA.dr}$

$dW = {}^1/_2\,\mathbf{D.E}.dv$

Energy = f (**D**,**E**) (6)



$dW = {}^1/_2\ \mathbf{D.E.}d\mathbf{A}.d\mathbf{r}$

$dW = {}^1/_2\ \mathbf{D.E.}dv$

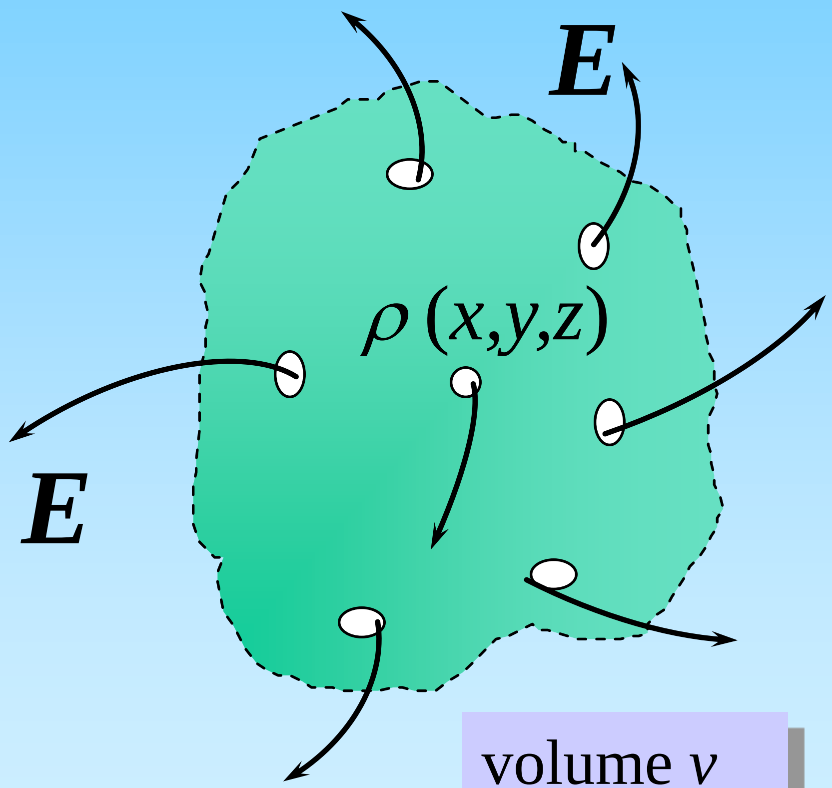
"Blow up" the sphere:
(all charges to infinity)

$$W = \frac{1}{2}\iiint_V \mathbf{D} \bullet \mathbf{E}\, dv$$

Conclusions

Following expressions were derived:

(ρ_f = density of free charges ; V = potential function)

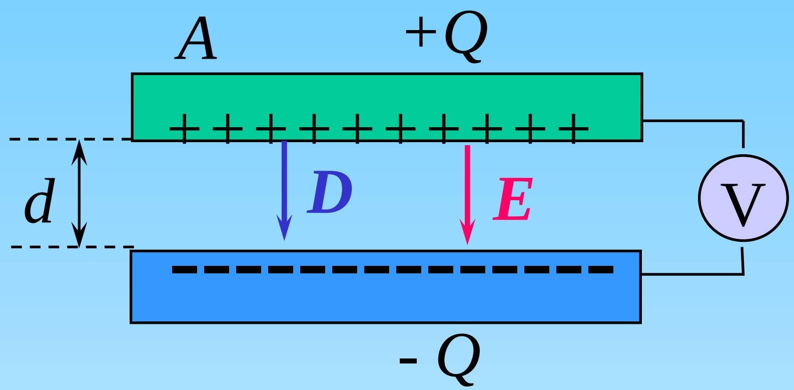


volume v
surface A

A. $W = \frac{1}{2} \iiint_V \rho_f . V \, dv$

B. $W = \frac{1}{2} \iiint_V \mathbf{D} \bullet \mathbf{E} \, dv$

Example: Parallel-plate capacitor



A. $W = \frac{1}{2}\iiint_V \rho_f . V \; dv$

$$W = \frac{1}{2} V \iint_A \sigma_f \, dA = \frac{1}{2} V Q_f = \frac{1}{2} C V^2$$

B. $W = \frac{1}{2}\iiint_V \mathbf{D} \bullet \mathbf{E} \; dv$

Gauss: $D = \sigma_f$

$E = V/d$

$C = Q/V$

$$W = \frac{1}{2}\iiint_{vol} \sigma_f \frac{V}{d} dv = \frac{1}{2} \sigma_f \frac{V}{d} A.d = \frac{1}{2} \sigma_f VA.d = \frac{1}{2} Q_f V = \frac{1}{2} C V^2$$

the end