The Electric Field of a Dipole

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## Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss’ Law for a cylindrical charge
- Gauss’ Law for a charged plane
- Laplace’s and Poisson’s Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere
The Electric Field of a Dipole

Given:
Electric Dipole, situated in O, with dipole moment $p$ [Cm]

Question:
Calculate $E$-field in arbitrary points $P$ around the dipole
The Electric Field of a Dipole

- Analysis and symmetry
- Approach to solution
- Calculations
- Conclusions
- Field lines in XY-plane
Analysis and Symmetry

Coordinate axes: X, Y, Z
Y-axis = dipolar axis

Symmetry: cylindrical around Y-axis

Cylindrical coordinates:
\( r, \theta \) and \( \phi \) (perpendicular')

All points at equal \( r \) and \( \theta \) are equivalent, even if at different \( \phi \)
Approach to solution

Several approaches possible:

1. $E$ with “Coulomb”;
2. $E$ from $V$ with:
   \[ E = - \text{grad} \, V \]

Choose option 2.

Necessary: calculate $V(r, \theta, \varphi)$ first!
Intermezzo: $V(r, \theta, \phi)$ - calculation

Calculate: Potential in $P$ : (reference in inf.)

Contributions to $V$ from $+Q$ and $-Q$ :

$$V(P) = \frac{+Q}{4\pi\varepsilon_0 r_+} + \frac{-Q}{4\pi\varepsilon_0 r_-}$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{r_+ - r_-}{r_+ r_-}$$

If $a \ll r_+, r_-$, then

$$V(r, \theta) = \frac{Q}{4\pi\varepsilon_0} \frac{a \cos \theta}{r^2}$$

$p = Qa$
Intermezzo: $V(r, \theta, \varphi)$ - calculation

$V(r, \theta) = \frac{Q}{4\pi\varepsilon_0} \frac{a \cos \theta}{r^2} = \frac{p \cos \theta}{4\pi\varepsilon_0 r^2}$

This approximation is called:

Far-field approximation
Calculation of $E$ (1)

Calculate $E$ from $V$ using:

$$E = - \nabla V$$

$$E = -\frac{\partial V}{\partial x}e_x - \frac{\partial V}{\partial y}e_y - \frac{\partial V}{\partial z}e_z$$

In general:

$$E = -\frac{\partial V}{\partial s_u}e_u - \frac{\partial V}{\partial s_v}e_v - \frac{\partial V}{\partial s_w}e_w$$

with $s_u$, $s_v$ and $s_w$ line elements
Calculation of $E_{(2)}$

\[ E = - \frac{\partial V}{\partial s_u} e_u - \frac{\partial V}{\partial s_v} e_v - \frac{\partial V}{\partial s_w} e_w \]

Here \( u=r, \ v=\theta, \ w=\varphi \)

\[ \partial s_u = \partial r \]

With \( V(r, \theta) = \frac{p \cos \theta}{4\pi \varepsilon_0 r^2} \)

\[ E_r = -\frac{\partial V}{\partial r} e_r = \frac{2p \cos \theta}{4\pi \varepsilon_0 r^3} e_r \]
Calculation of $E(3)$

$E = - \frac{\partial V}{\partial s_u} e_u - \frac{\partial V}{\partial s_v} e_v - \frac{\partial V}{\partial s_w} e_w$

here $u=r$, $v=\theta$, $w=\phi$

$\partial s_v = r \partial \theta$

with $V(r,\theta) = \frac{p \cos \theta}{4\pi \varepsilon_0 r^2}$

$E_{\theta} = -\frac{\partial V}{r \partial \theta} e_{\theta} = \frac{p \sin \theta}{4\pi \varepsilon_0 r^3} e_{\theta}$
Calculation of $\mathbf{E}$ (4)

$E = - \frac{\partial V}{\partial s_u} e_u - \frac{\partial V}{\partial s_v} e_v - \frac{\partial V}{\partial s_w} e_w$

Here $u=r$, $v=\theta$, $w=\varphi$

$\partial s_v = r \sin \theta \partial \varphi$

With $V(r, \theta) = \frac{p \cos \theta}{4\pi\varepsilon_0 r^2}$

$E_\varphi = - \frac{\partial V}{r \sin \theta \partial \varphi} e_\varphi = 0 e_\varphi$
Calculation of $E$ (5)

\[
E_r = \frac{2p \cos \theta}{4\pi \varepsilon_0 r^3} e_r
\]

\[
E_\theta = \frac{p \sin \theta}{4\pi \varepsilon_0 r^3} e_\theta
\]

\[
E_\varphi = 0 e_\varphi
\]
Conclusions (1)

\[ E = \frac{p}{4\pi \varepsilon_0 r^3} \left( 2\cos\theta e_r + \sin\theta e_\theta \right) \]

\[ V(r, \theta) = \frac{p \cos\theta}{4\pi \varepsilon_0 r^2} \]

\[ V(r, \theta) = \frac{p \cdot e_r}{4\pi \varepsilon_0 r^2} \]
Conclusions (2)

Field of a dipole

\[ E = \frac{p}{4\pi \varepsilon_0 r^3} \left( 2\cos\theta e_r + \sin\theta e_\theta \right) \]

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E\(_r\) and E\(_\theta\) - comp.

E-field of a dipole
Field lines in XY-plane (1)

Field lines in XY-plane:

\[ \frac{dy}{dx} = \frac{E_y}{E_x} \]

E-field of a dipole

\[ E_r = \frac{2p \cos \theta}{4\pi\varepsilon_0 r^3} e_r \]

\[ E_\theta = \frac{p \sin \theta}{4\pi\varepsilon_0 r^3} e_\theta \]
Field lines in $XY$-plane (2)

Parameter: distance $a$ between $-Q$ and $+Q$

- $a = 20$
- $a = 10$
- $a = 1$
- $a = 0.1$

far field
Field lines in XY-plane (3)

Parameter: distance $a$ between $-Q$ and $+Q$

- $a = 20$
- $a = 10$
- $a = 1$
- $a = 0.1$

far field
Field lines in XY-plane (4)

Parameter: distance $a$ between -Q and +Q

- $a = 20$
- $a = 10$
- $a = 1$
- $a = 0.1$

far field
Field lines in XY-plane (5)

Parameter: distance $a$ between -Q and +Q

$a = 20$
$a = 10$
$a = 1$
$a = 0.1$
far field
Field lines in XY-plane (6)

Parameter: distance $a$ between -Q and +Q

- $a = 20$
- $a = 10$
- $a = 1$
- $a = 0.1$

far field
Field lines in XY-plane (7)

Parameter: distance $a$ between $-Q$ and $+Q$

$E$-field of a dipole

the end