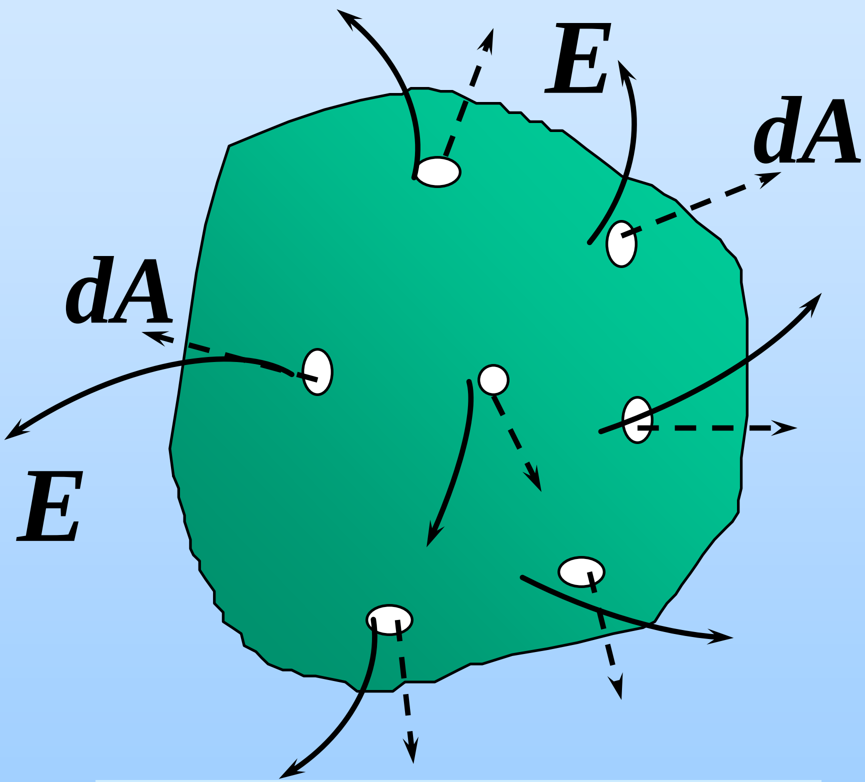
# The Divergence Theorem and Electric Fields

# Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

# Gauss' Law for *E*-field (1)




volume *V*; surface *A*

***dA*** ⊥ surface *A*

***E*-field arbitrary**

**Gauss' Law:**

$$\oiint_A \mathbf{E}\bullet d\mathbf{A} = \frac{Q}{\varepsilon_0} = \iiint_V \frac{\rho}{\varepsilon_0}\, dV$$

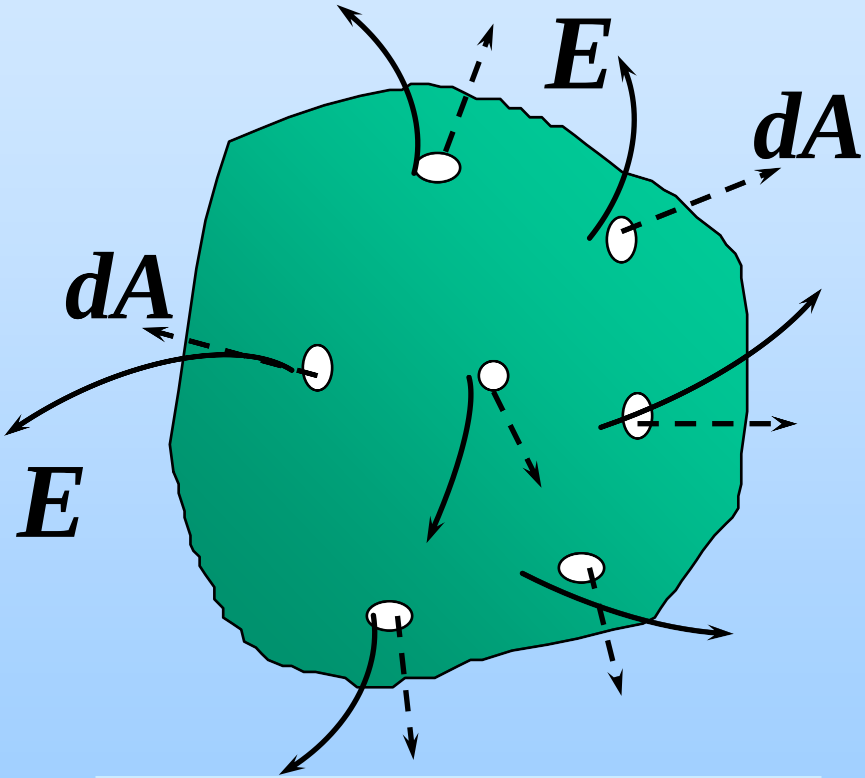
Goal of this integral expression :

to calculate ***E*** from *Q* or *ρ*,

provided symmetry present !!!!!

QUESTION: does an inverse expression, to <u>locally</u> calculate *ρ(xyz)* from ***E****(xyz)*, exist ??

# Gauss' Law for *E*-field (2)




volume *V*; surface *A*

***dA*** ⊥ surface *A*

***E***-field arbitrary

**<u>Gauss' Law:</u>**

$$\oint\!\!\oint_A \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{Q}{\varepsilon_0} = \iiint_V \frac{\rho}{\varepsilon_0}\, dV$$

Gauss'Law is a *DIMENSION SWITCH*

It relates what happens inside the *volume*

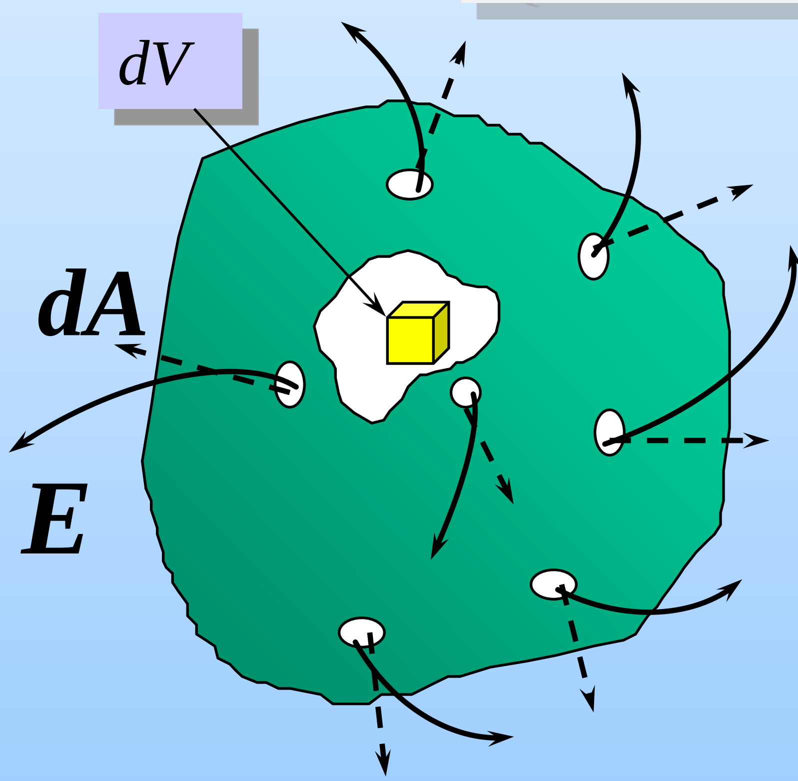
to what you observe at the *surface*

Other dimension switch relation:

*Stokes' Law for magnetic fields*

# Gauss' Law for ***E***-field (3)

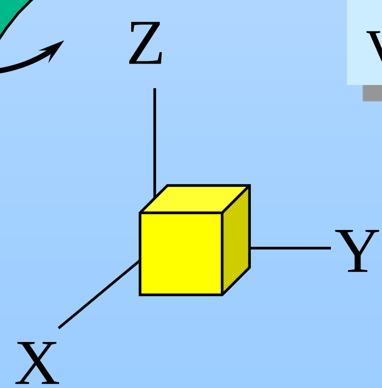
Question: calculate local $\rho$-distribution from ***E***-field




**<u>Gauss' Law:</u>**

$$\oint\!\!\oint_A \boldsymbol{E} \bullet d\boldsymbol{A} = \frac{Q}{\varepsilon_0} = \iiint_V \frac{\rho}{\varepsilon_0}\, dv$$

<u>to look locally:</u> observe local volume element *dV* at *(xyz)*

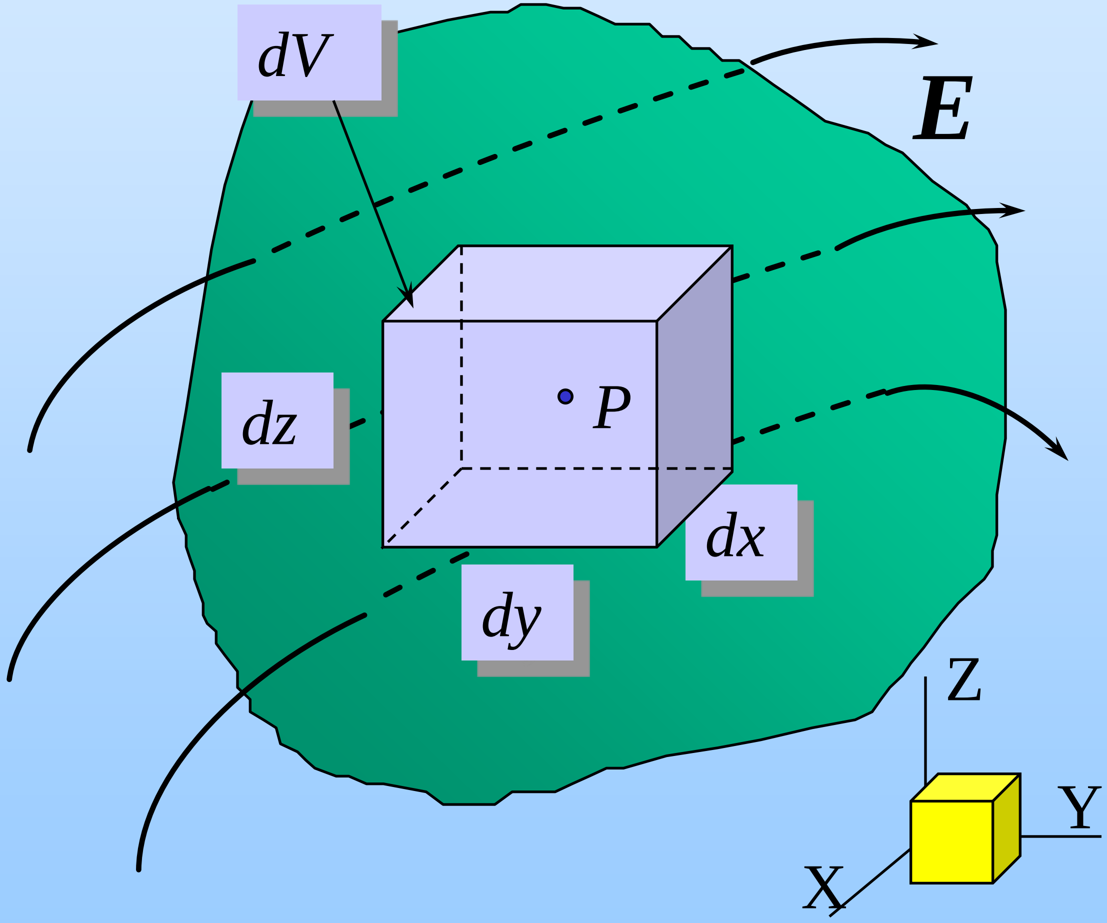



volume element *dV* has sides *dx*, *dy* and *dz*

volume *V*

surface *A*

# The ***E***-flux through *dV* (1)




Point $P$ in $dV$ at $(x,y,z)$.

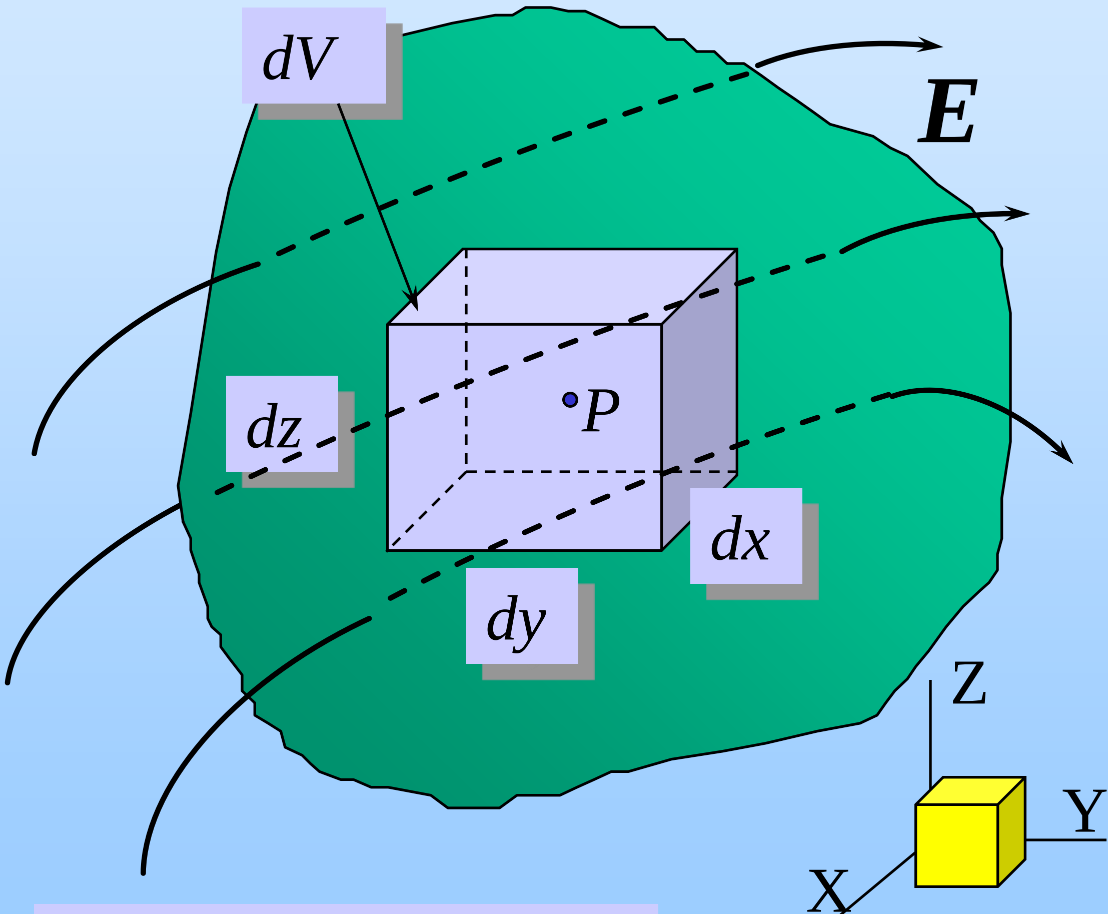
$d\Phi_E$ through $dV$ =

$d\Phi_E$ through left & right sides +

$d\Phi_E$ through top & bottom sides +

$d\Phi_E$ through front & back sides

# The $\boldsymbol{E}$-flux through $dV$ (2)




Point $P$ in $dV$ at $x,y,z$.

Point P is at $(x,y,z)$

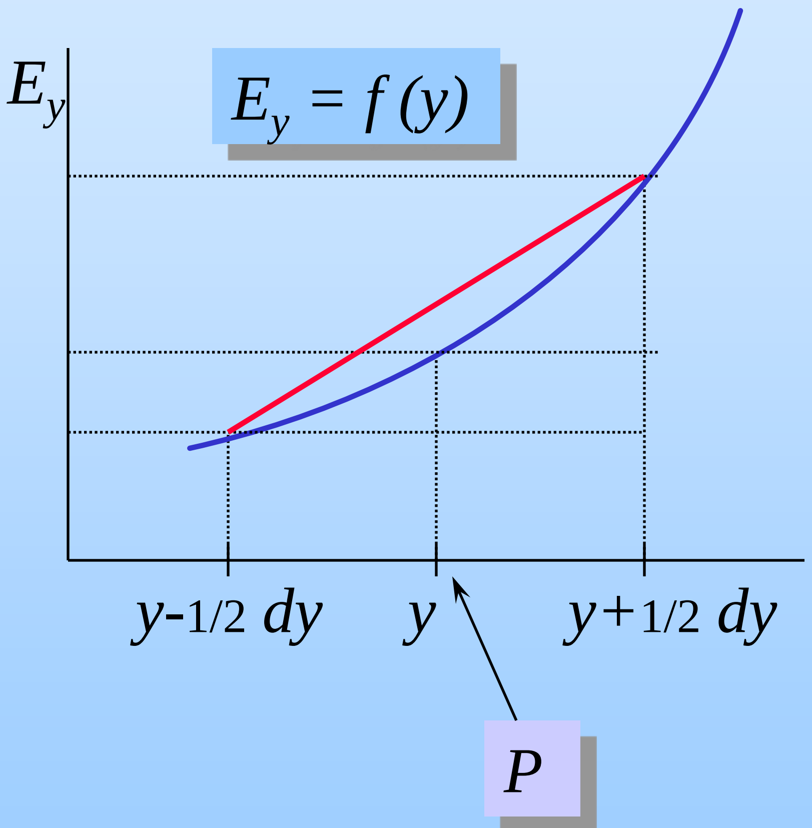
Calculate outward $d\Phi_E$ through *right* side:

$$d\Phi_E = \mathbf{E.dA} = E_y\, .dxdz$$
at $(x, y+dy/2, z)$

Calculate outward $d\Phi_E$ through *left* side:

$$d\Phi_E = \mathbf{E.dA} = -\, E_y\, .dxdz$$
at $(x, y - dy/2, z)$

# The ***E***-flux through *dV* (3)




Net outward flux $d\Phi_E$ through *left - right* side of *dV* :

$$d\Phi_E = E_y\,.dxdz \;\text{ at } (x, y + dy/2, z)$$

$$- E_y\,.dxdz \;\text{ at } (x, y - dy/2, z)$$

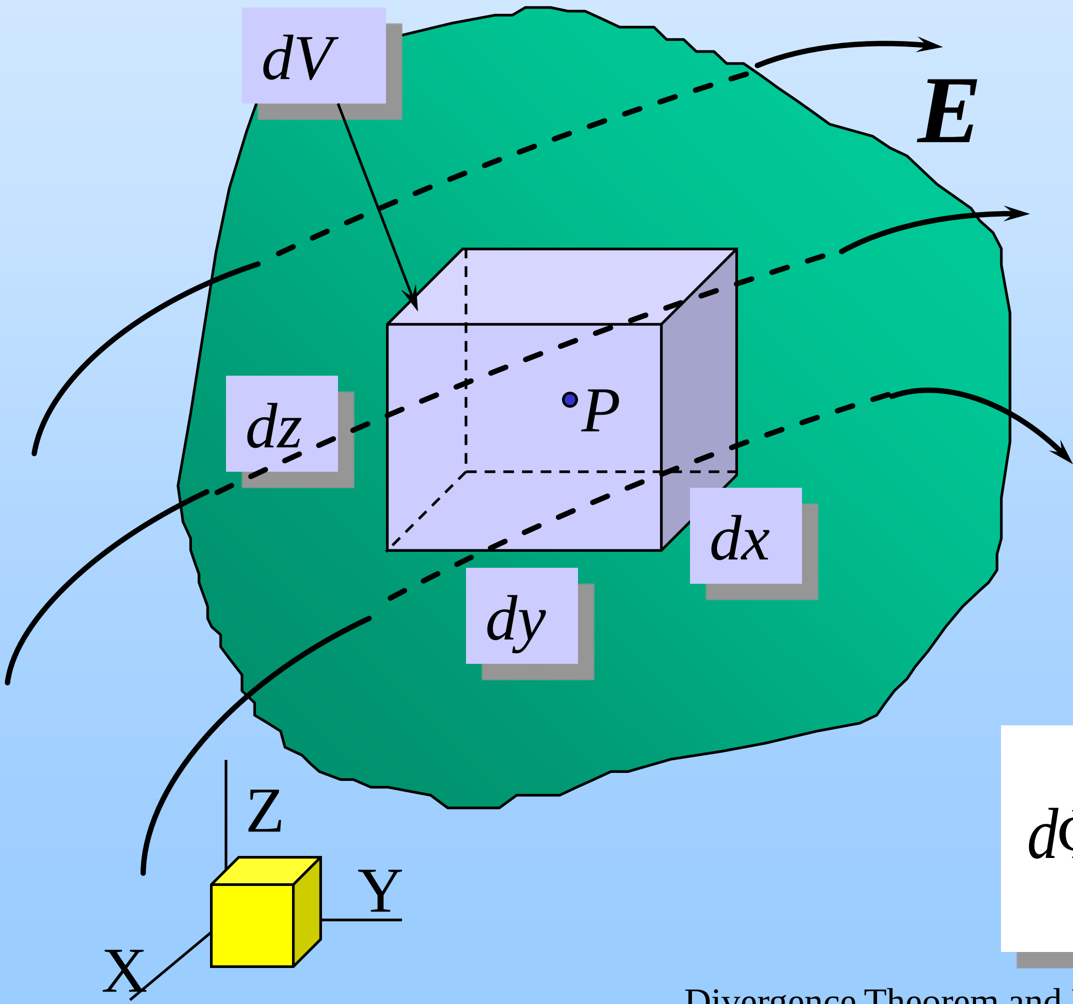
$$d\Phi_E = \left( E_y + \frac{dE_y}{dy}\frac{1}{2}dy \right) dx\,dz$$

$$- \left( E_y - \frac{dE_y}{dy}\frac{1}{2}dy \right) dx\,dz$$

Point *P* in *dV* at *x,y,z.*

$$\text{result}: d\Phi_E = \frac{dE_y}{dy} dxdydz = \frac{dE_y}{dy} dV$$

# The $\boldsymbol{E}$-flux through $dV$ (4)




Net outward flux $d\Phi_E$ :

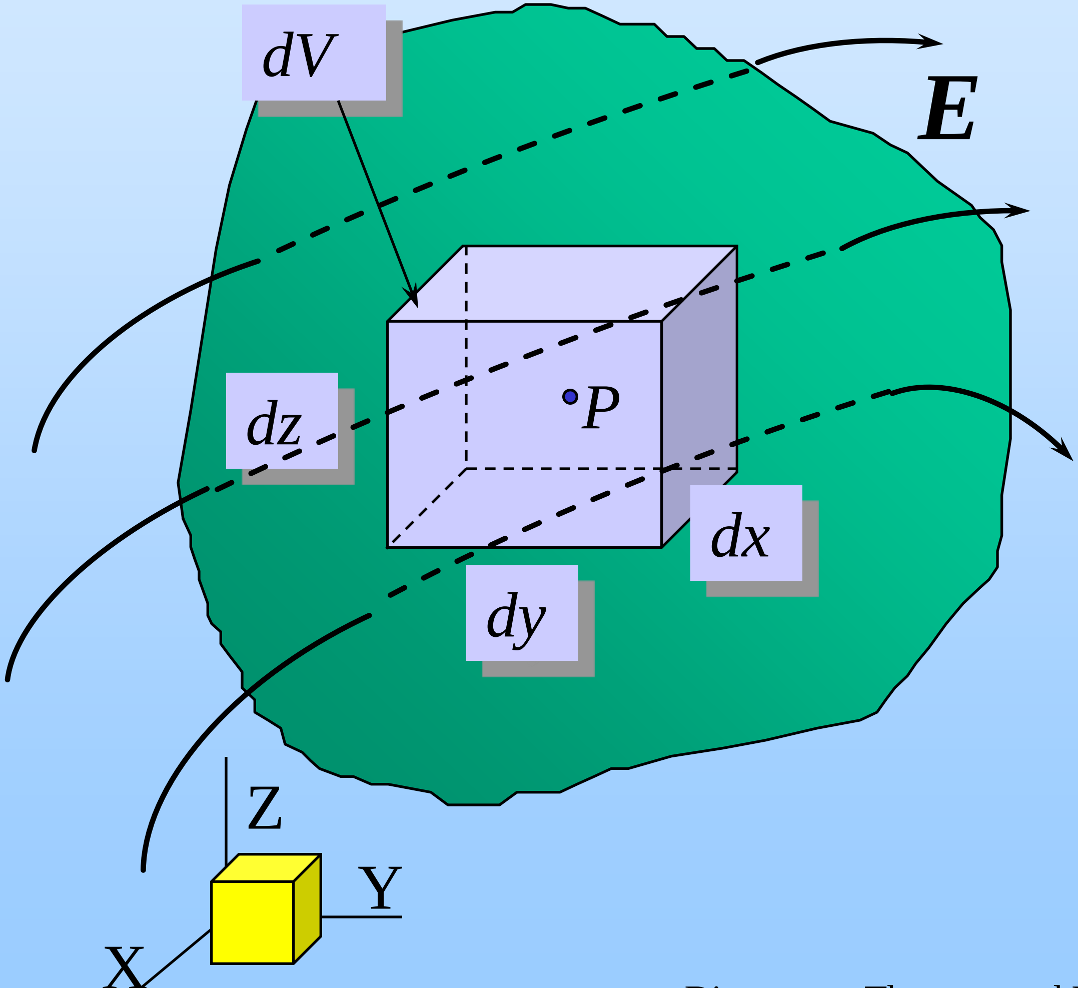
left/right: $\dfrac{dE_y}{dy} dV$

analogously:
top/bottom: $\dfrac{dE_z}{dz} dV$

and
front/back: $\dfrac{dE_x}{dx} dV$

$$d\Phi_E = \left[ \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} \right] dV$$

# The *E*-flux through *dV* (5)




Net outward flux $d\Phi_E$ through $dV$:

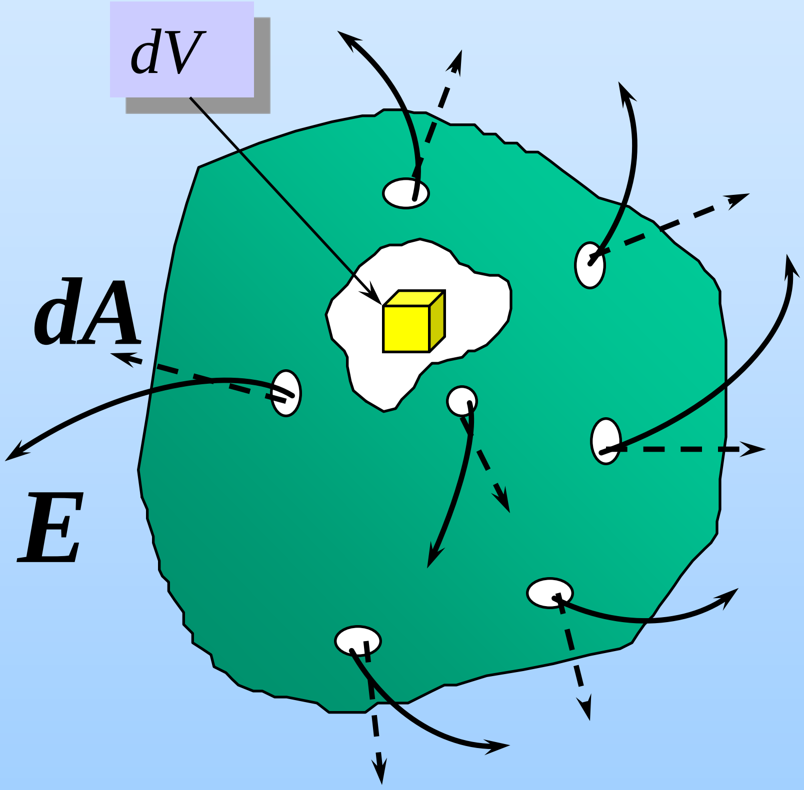
$$d\Phi_E = \left[ \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} \right] dV$$

**DEFINITION of DIVERGENCE *div* :**

$$\frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} \equiv div\ \mathbf{E}$$

$$\Rightarrow \quad d\Phi_E = div\ \mathbf{E}\ .dV$$

# Local expression for Gauss' Law




volume $V$

surface $A$

element $dV$ : $d\Phi_E = div\ \boldsymbol{E}\ .dV$

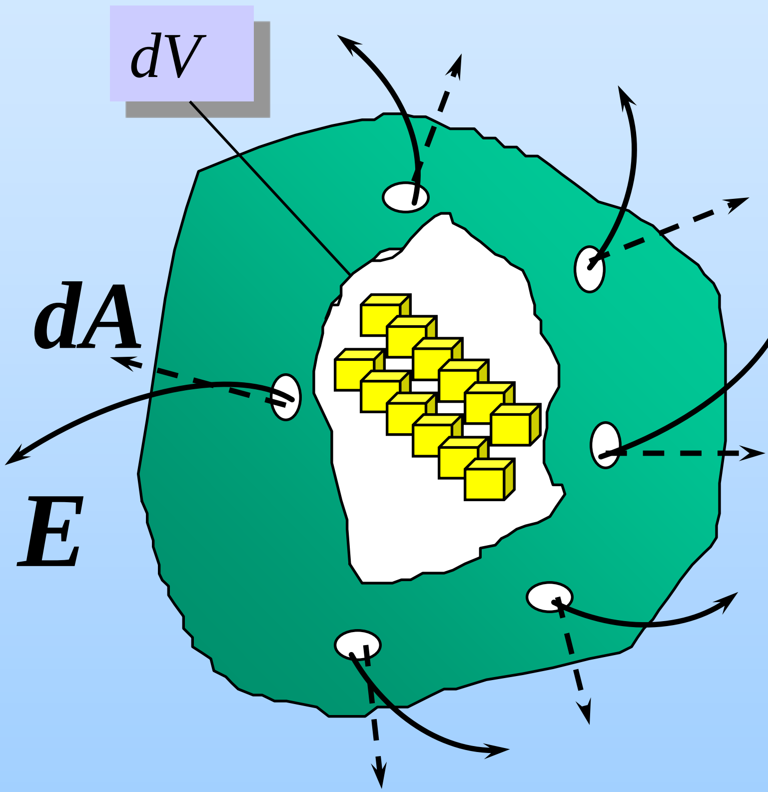
enclosed charge in $dV$ : $\rho\ .\ dV$

**Gauss' Law in local form:**

$$div\ \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

where $\boldsymbol{E}$ and $\rho$ are $f$ (x,y,z)

# From “local” to “integral”



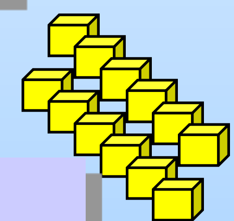

volume $V$

surface $A$

element $dV$ :

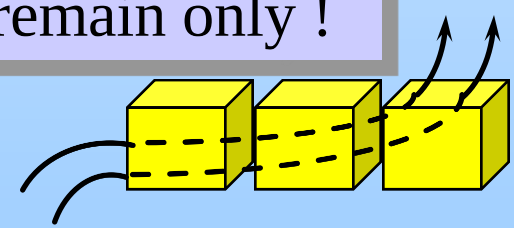
$$d\Phi_E = div\ \ \boldsymbol{E}\ .dV = \frac{\rho}{\varepsilon_0} dV$$

summation over all elements:



all “internal” $d\Phi_E$’s cancel

$d\Phi_E$’s at surface $A$ remain only !

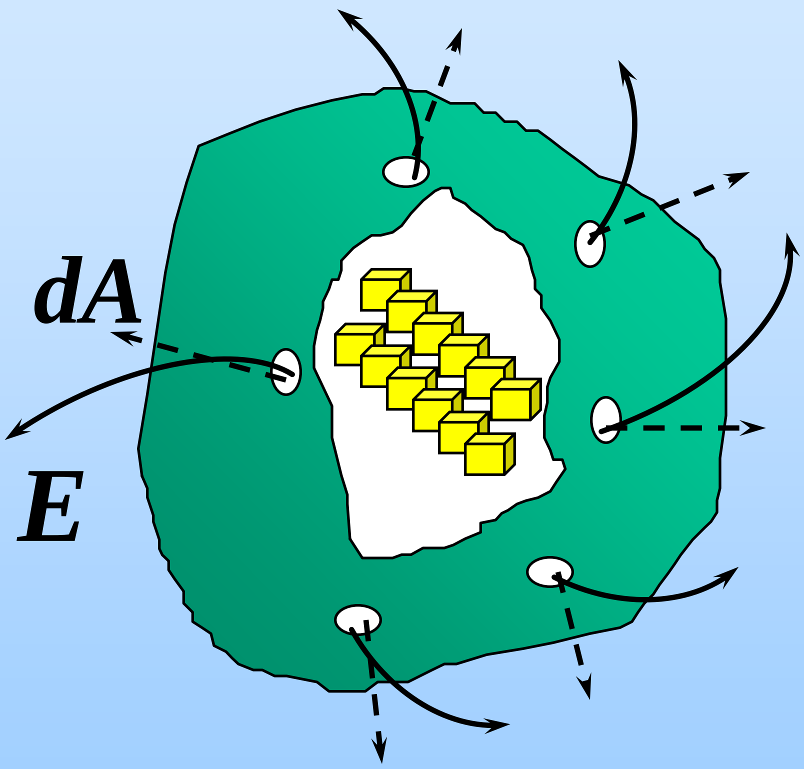


$$\Phi_E = \oiint_A \boldsymbol{E} \bullet \boldsymbol{dA}$$

$$\Phi_E = \iiint_V div\ \boldsymbol{E}\ dV = \iiint_V \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

# How to use the laws ?

$$\Phi_E = \oiint_A \boldsymbol{E} \bullet \boldsymbol{dA} = \frac{Q}{\varepsilon_0} = \iiint_V \frac{\rho}{\varepsilon_0} dV$$




Integral expression:

calculate ***E*** from $\rho$**,** but in symmetrical situations only !

div ***E*** (x,y,z) = $\rho$ (x,y,z) / $\varepsilon_0$

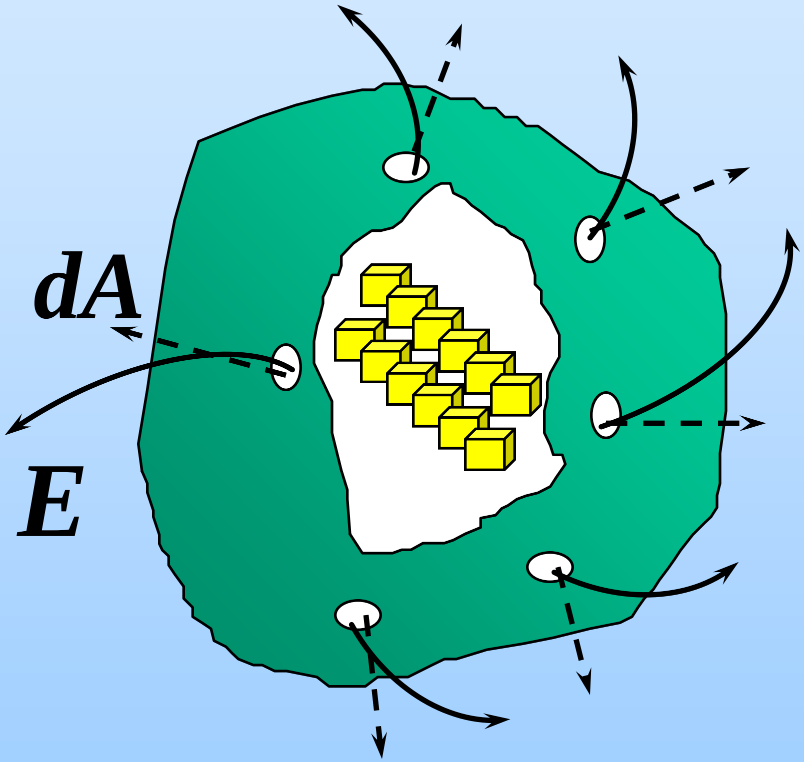
volume *V*

surface *A*

Differential (local) expression:

calculate $\rho$ (x,y,z) from ***E*** (x,y,z).

# Physical meaning of *div*




volume $V$

surface $A$

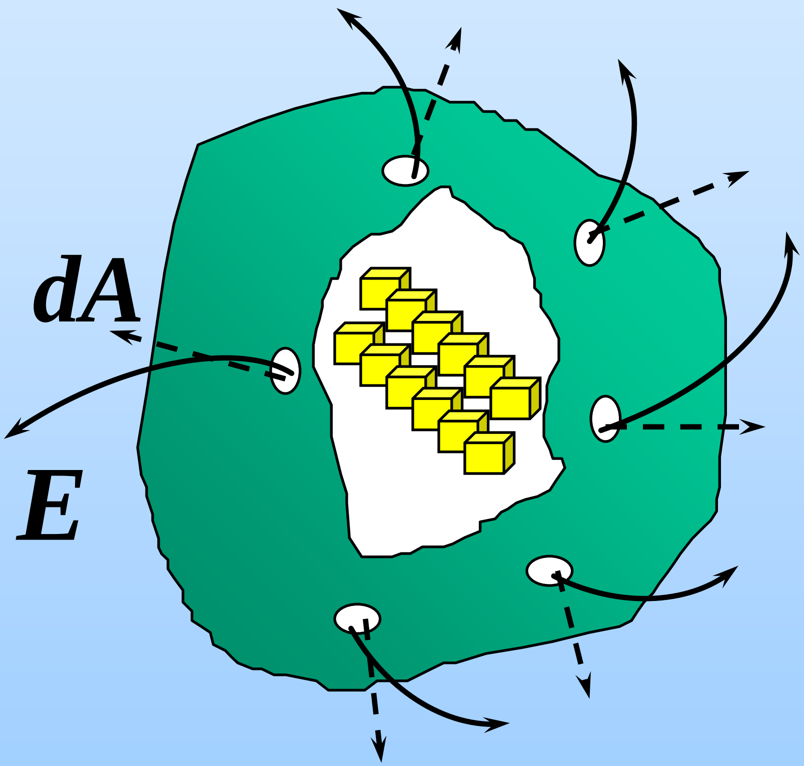
$$\Phi_E = \oint\!\!\oint_A \mathbf{E} \bullet \mathbf{dA} =$$

$$= \iiint_V div\, \mathbf{E}\, dV = \iiint_V \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

$$d\Phi_E = div\, \mathbf{E}\, .dv \Rightarrow div\, \mathbf{E} = \frac{d\Phi_E}{dv}$$

Divergence = local "micro"-flux per unit of volume [$m^3$]

# “Gauss” in general




volume $V$

surface $A$

$$\Phi_E = \oint\!\!\oint_A \boldsymbol{E} \bullet \boldsymbol{dA} =$$

$$= \iiint_V div\ \boldsymbol{E}\ dV = \iiint_V \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

in accordance with general relation for a vector $\boldsymbol{X}$ :

$$\oint\!\!\oint_A \boldsymbol{X}.\boldsymbol{dA} = \iiint_V div\ \boldsymbol{X}\ dV$$

The Divergence Theorem is used in many fields of physics



*the end*