Magnetic Field of a Rotating Charged Conducting Sphere

2nd version: on-axis and off-axis

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Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

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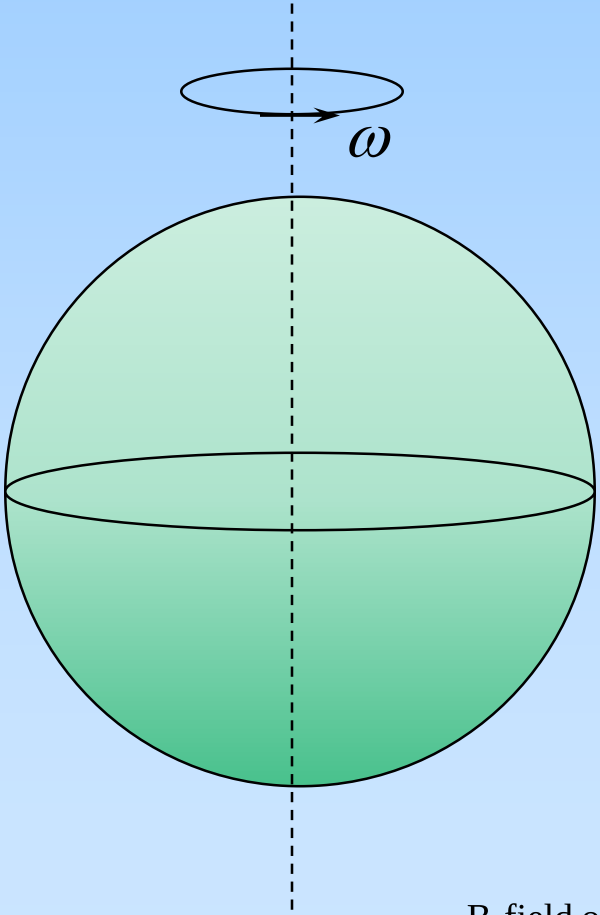
B-field of a rotating charged <u>conducting</u> sphere



<u>Available:</u>

A charged conducting sphere (charge Q, radius R), rotating with ω rad/sec

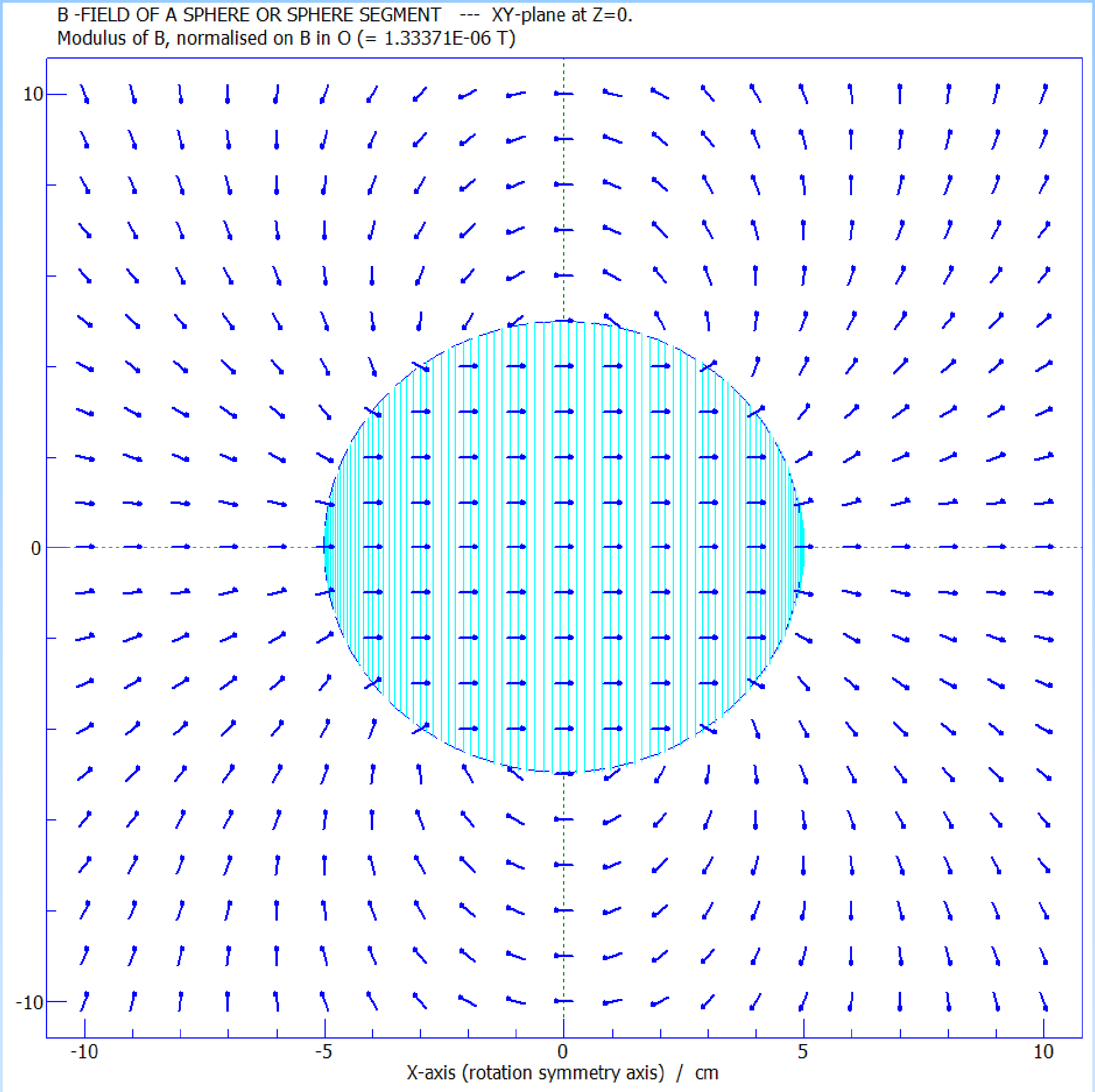
<u>Question:</u>

Calculate ***B***-field in arbitrary points inside and outside the sphere

I. on the axis of rotation

II. off-axis

Ad. I : analytical approach possible
Ad. II : numerical approach needed

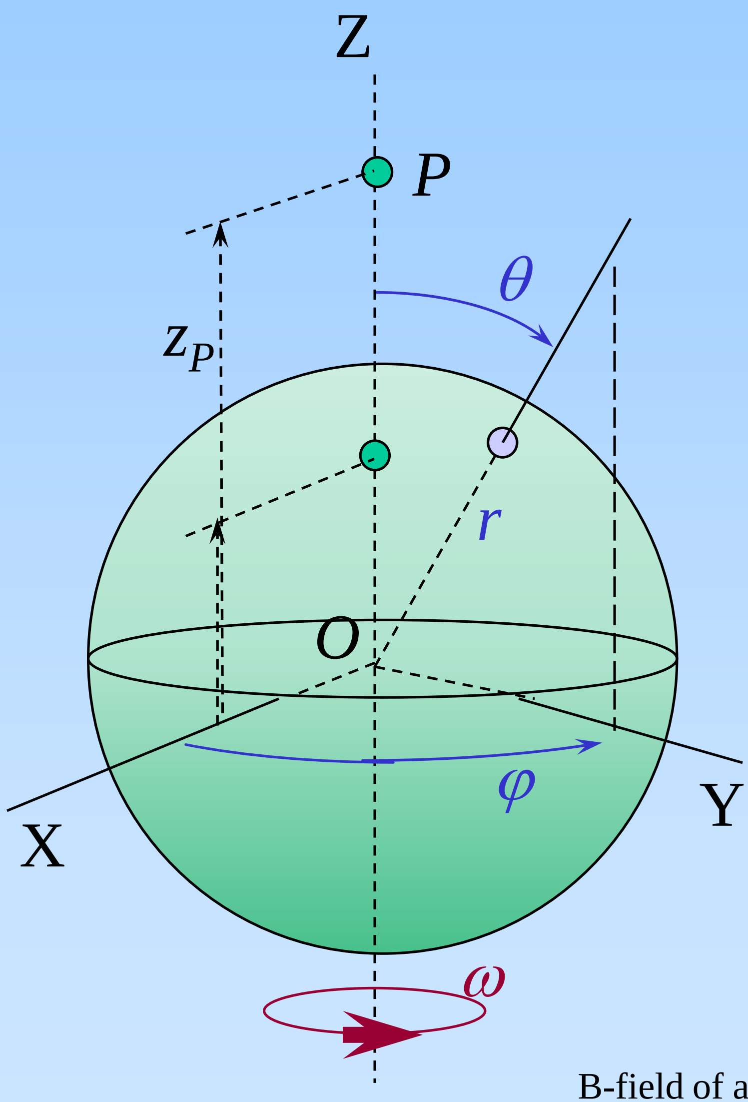


Objective:

B-field:
of a
charged conductive sphere
rotating around the X-axis

Inside the sphere:
homogeneous field

Analysis and Symmetry for on-axis (1)



Assume P on Z-axis.

Part I. Calculate ***B***-field in point *P*

on the axis of rotation (Z-axis)

inside or outside the sphere

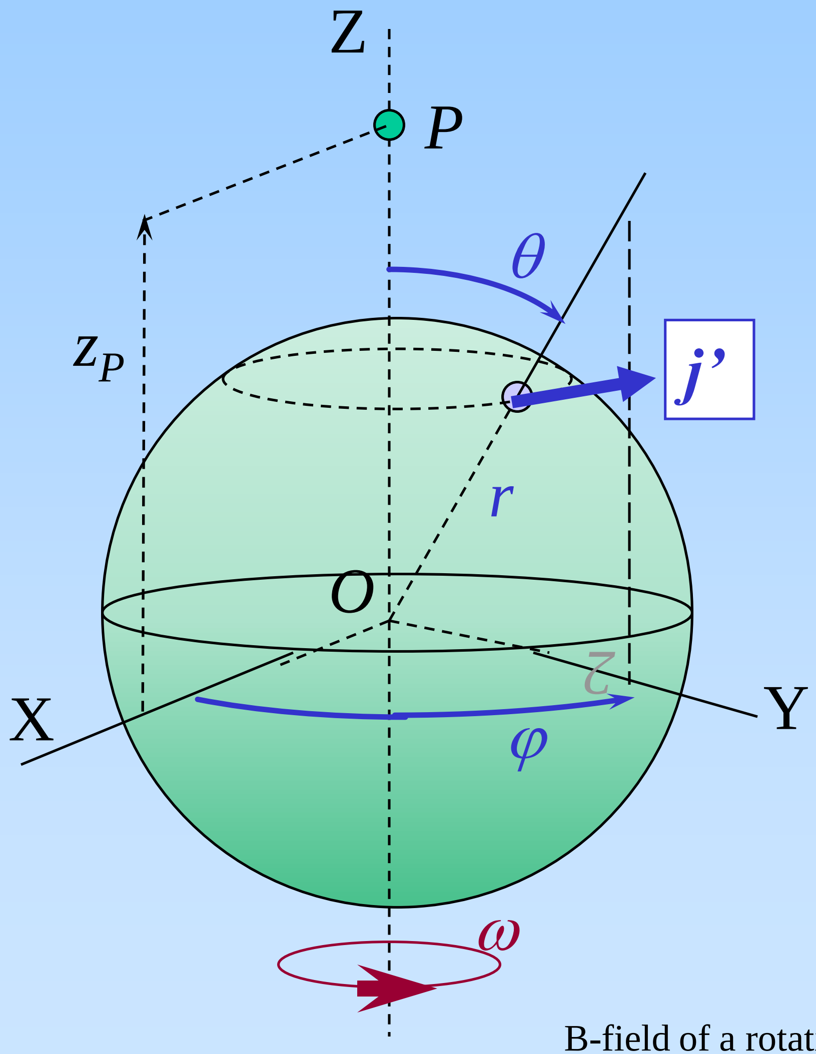
(Part II : points P *off-axis*)

Coordinate systems:

- X,Y, Z
- r, θ, φ

Symmetry: around rotation axis.

Analysis and Symmetry for on-axis (2)



<u>Conducting sphere</u>,

all charges at surface:

surface charge density:

$\sigma = Q/(4\pi R^2)$ [C/m²]

Rotating charges will establish a "surface current",

directed along surface circles.

Surface current density ***j'*** [A/m]:

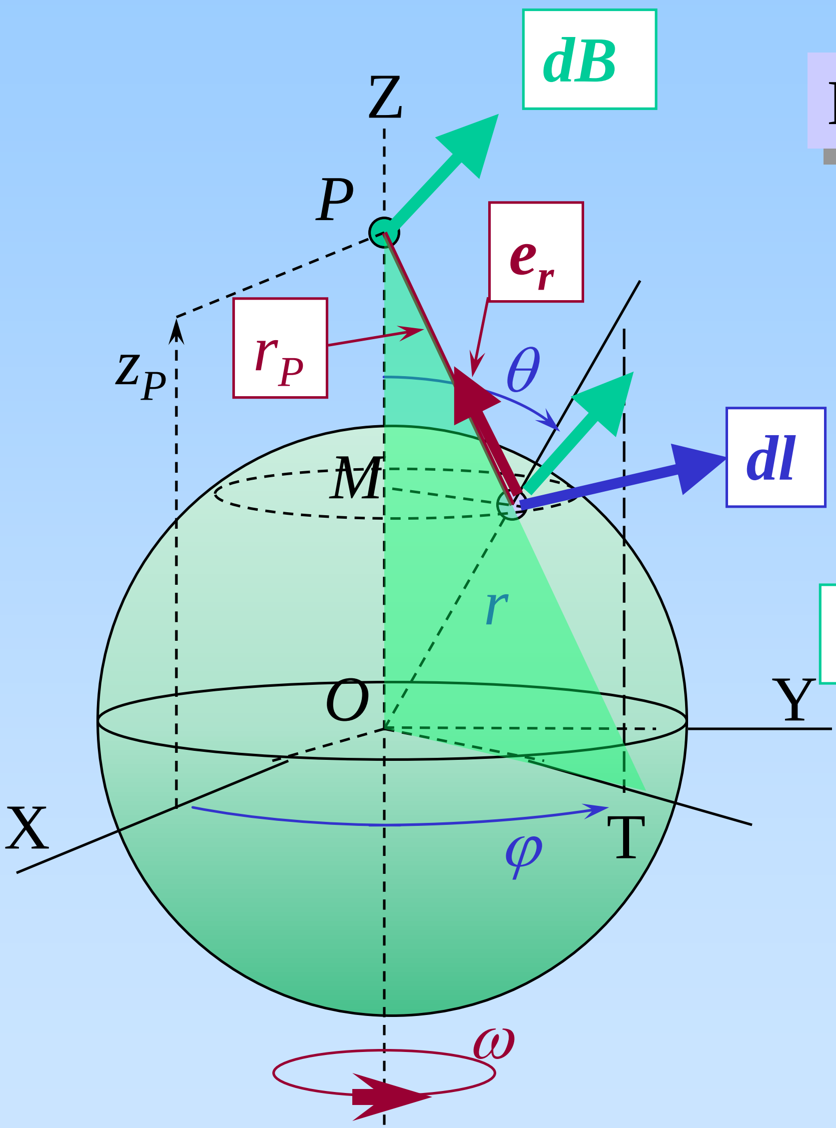
will be a function of θ

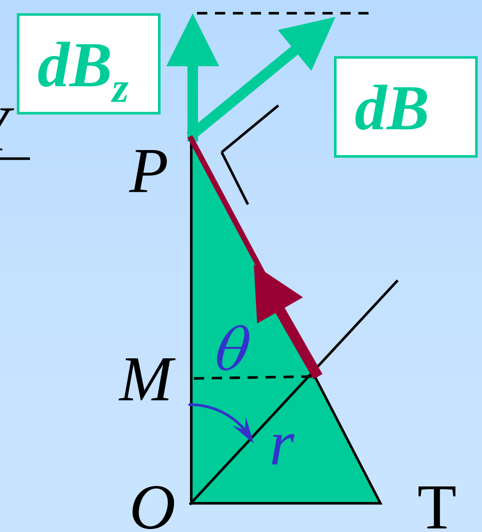
Analysis and Symmetry for on-axis (3)

Biot & Savart :

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\,\frac{I.\boldsymbol{dl}\times\boldsymbol{e_r}}{r_P{}^2}$$

$\boldsymbol{dB} \perp \boldsymbol{dl}$ and $\boldsymbol{e_r}$.
if P = on-axis: $\boldsymbol{dl} \perp \boldsymbol{e_r}$





B-field of a rotating charged conducting sphere

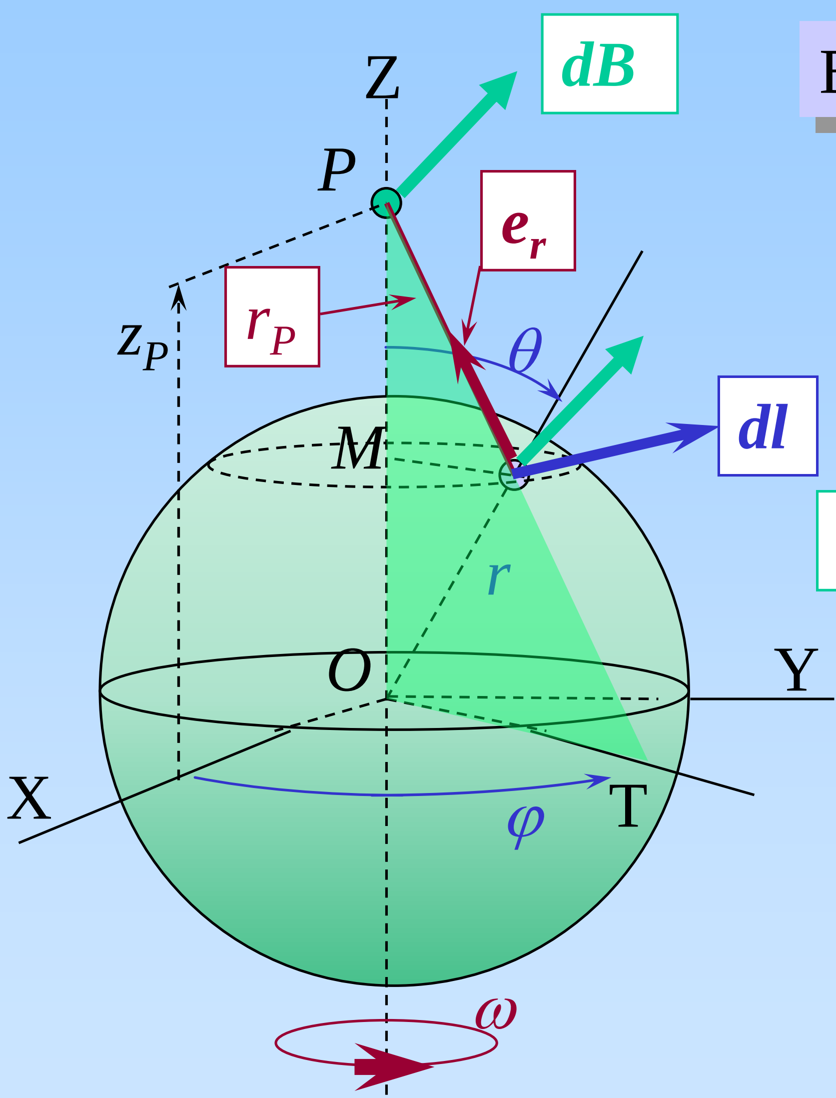
Direction of ***dB***:

Cylindrical symmetry
around Z-axis:

Z-comp. only !!

X- and Y-comp. cancel.

Analysis and Symmetry for on-axis (4)

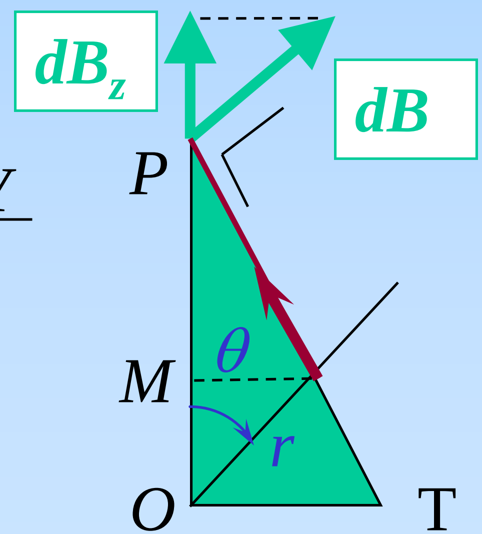


Biot & Savart :

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I.\boldsymbol{dl}\times\boldsymbol{e_r}}{r_P{}^2}$$

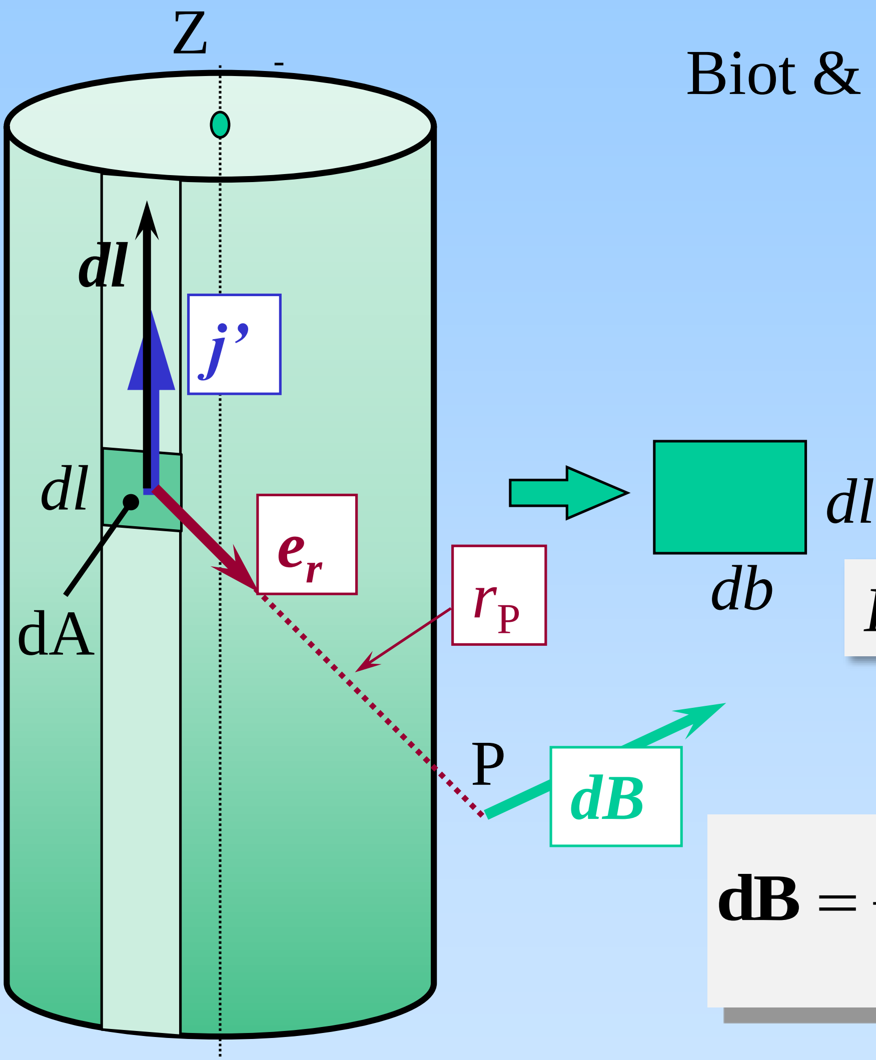
dB*, *dl and ***e***

$\boldsymbol{dB}$, $\boldsymbol{dl}$ and $\boldsymbol{e_r}$ mutually perpendicular



Question:
How to relate
I.dl (in A.m)
to surface current density j (in A/m^2)

Intermezzo: a surface current



Biot & Savart:

$$\mathbf{dB} = \frac{\mu_0}{4\pi}\frac{I.\mathbf{dl}\times\mathbf{e}_r}{r_P^{\,2}}$$

Current strip at surface:
j': current density[A/m]

NB. Density [A/m] =
current per m width!

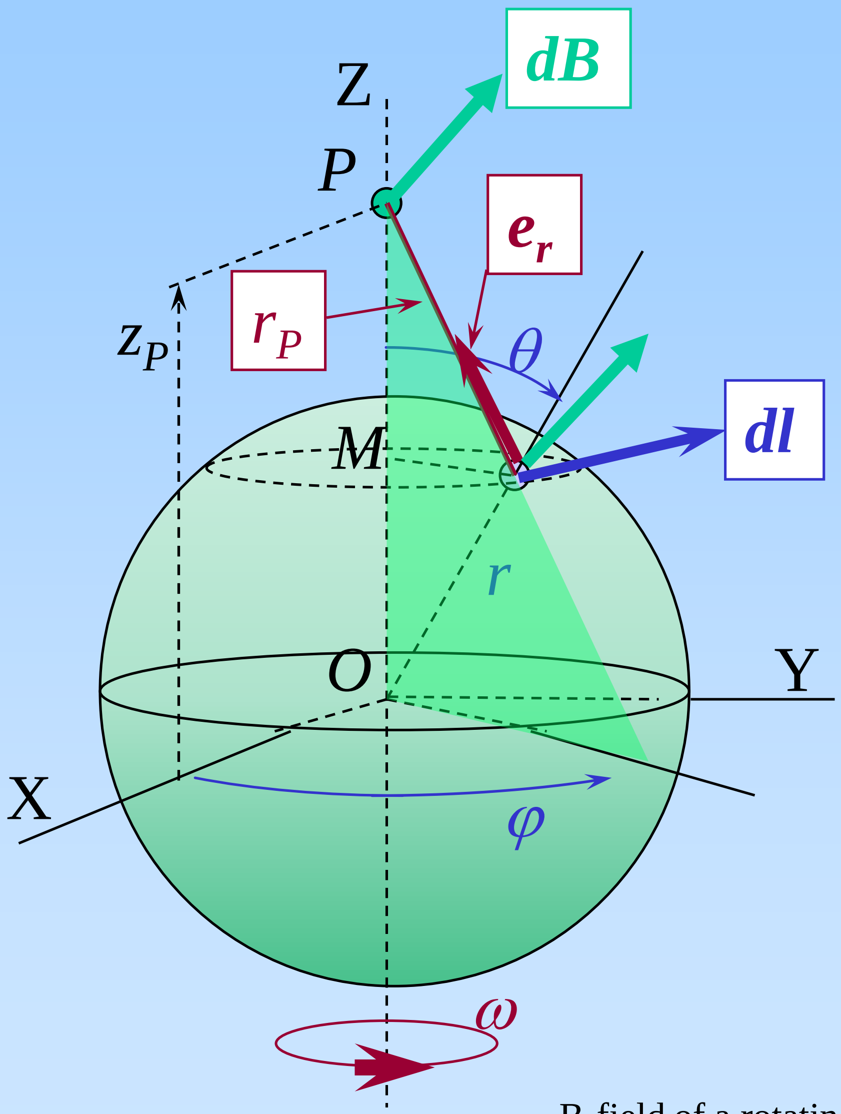
$I.\mathbf{dl} = j'.db.\mathbf{dl} = \mathbf{j'}.db.dl = \mathbf{j}'.dA$

Biot & Savart :

$$\mathbf{dB} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}'db.dl\times\mathbf{e_r}}{r_P^{\,2}} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}'\times\mathbf{e_r}}{r_P^{\,2}}dA$$

$\mathbf{dB} \perp \mathbf{dl}$ and $\mathbf{e}_r$

Analysis and Symmetry for on-axis (4)



Biot & Savart :

$$\mathbf{dB} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}'\,db.dl \times \mathbf{e_r}}{r_P^{\;2}} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}' \times \mathbf{e_r}}{r_P^{\;2}}\,dA$$

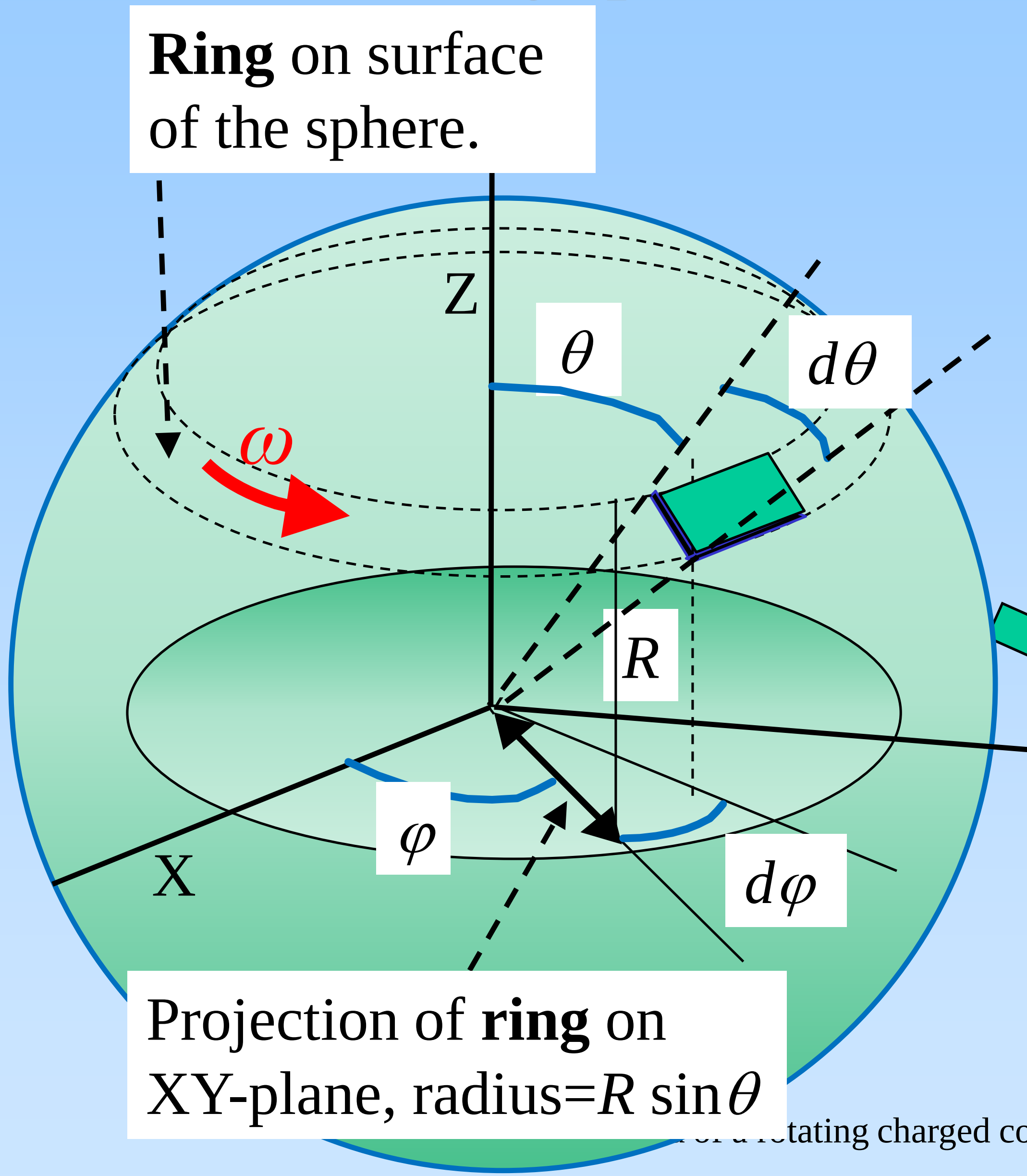
dB*, *dl and ***e*$_r$** mutually perpendicular

Needed:

expressions for:

dA , j', $\mathbf{e_r}$, r_P

Conducting sphere: on-axis (1)



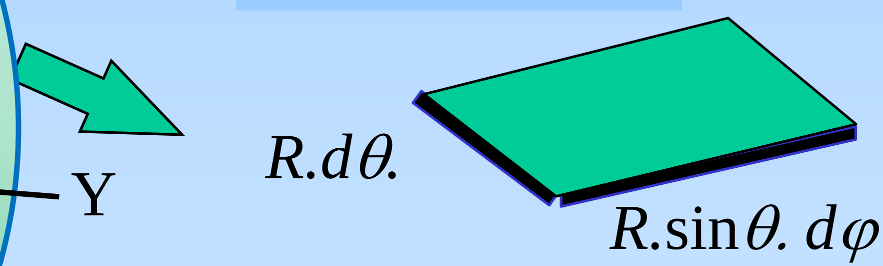
Needed: expressions for: dA , j', $\mathbf{e_r}$, r_P

Conducting sphere,

surface density:

$\sigma = Q/(4\pi R^2)$

Surface element:



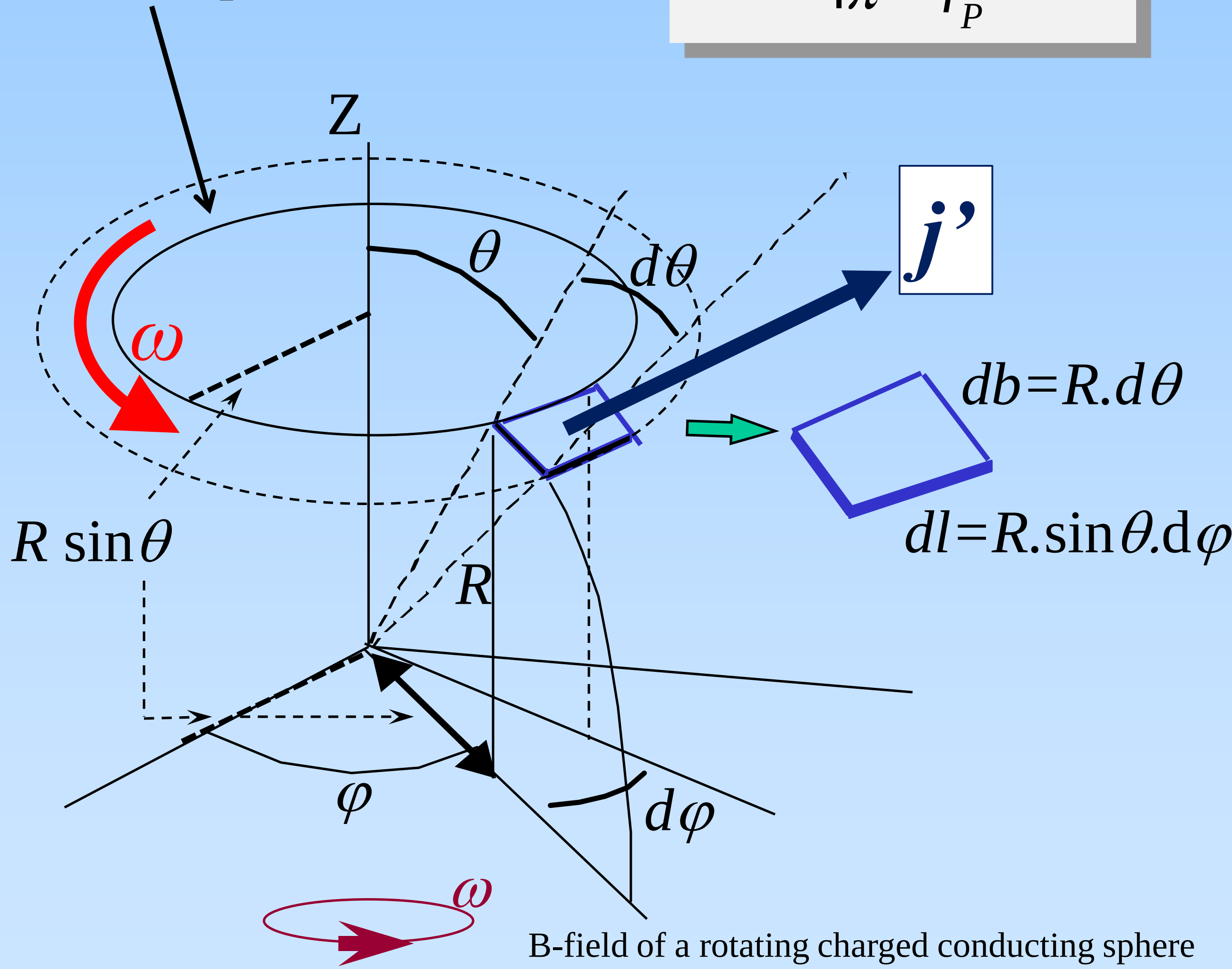
surface element:

$dA = (R.d\theta).(R.\sin\theta.\, d\varphi)$

Conducting sphere: on-axis (2)

Ring on surface of the sphere.

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j}' \times \boldsymbol{e_r}}{r_P^{\;2}}\, dA$$



Needed: expressions for:
dA, j', $\boldsymbol{e_r}$, r_P

with j' in [A/m]

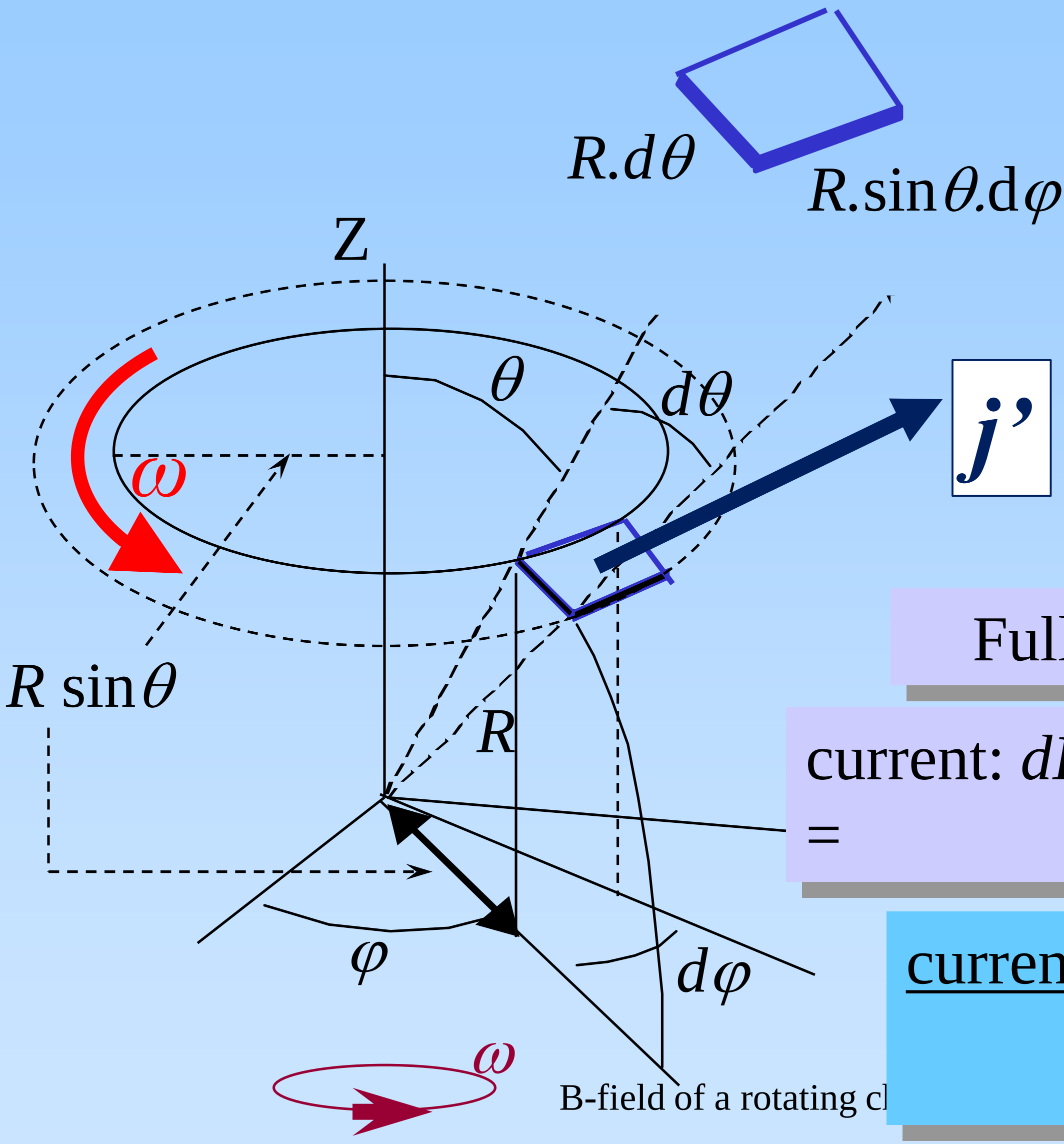
<u>Surface charge</u>
$\sigma.dA$ on dA will rotate with ω

$dA = db \bullet dl$

Conducting sphere: on-axis (3)

Needed:

dA, j', $\mathbf{e_r}$, r_P



B-field of a rotating cl

<u>Ring on surface</u>: area = $2\pi(R.\sin\theta).(Rd\theta)$

<u>Charge on ring</u> : $\sigma.\ 2\pi R.\sin\theta\ .\ Rd\theta$

Full rotation over 2π in $2\pi/\omega$ s.

current: $dI = \sigma.2\pi R.\sin\theta.\ Rd\theta\ /\ (2\pi/\omega)$

$= \sigma\,\omega\,R\,\sin\theta\ .\ Rd\theta$

<u>current density:</u> $j' = dI\ /\ (Rd\theta) =$

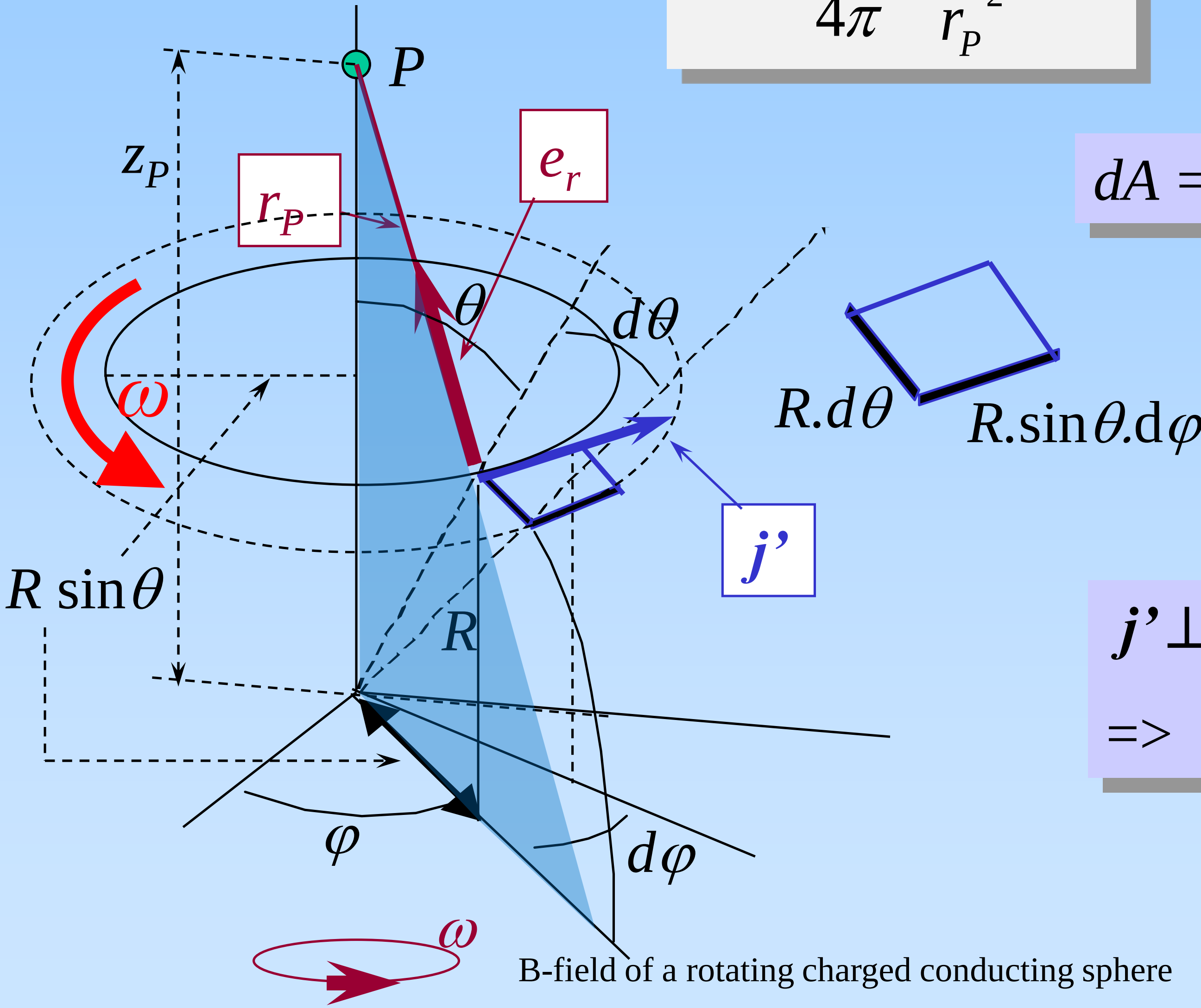
$j' = \sigma\,\omega\,R.\sin\theta$ [A/m]

Conducting sphere: on-axis (4)

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j}' \times \boldsymbol{e_r}}{r_P^{\,2}}\,dA$$

Needed:

$dA,\ j',\ \boldsymbol{e_r},\ r_P$



$dA = R.\sin\theta.\ d\varphi.R.d\theta.$

$j' = \sigma\omega R \sin\ \theta$

$\boldsymbol{j'} \perp \boldsymbol{e_r}$:

$\Rightarrow\ |\boldsymbol{j'} \times \boldsymbol{e_r}| = j'.|e_r| = j'$

Conducting sphere: on-axis (5)

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{j}' \times \boldsymbol{e_r}}{r_P^{\,2}} dA$$

Needed:

dA , j', $\boldsymbol{e_r}$, r_P

$j' = \sigma\omega R \sin\,\theta$

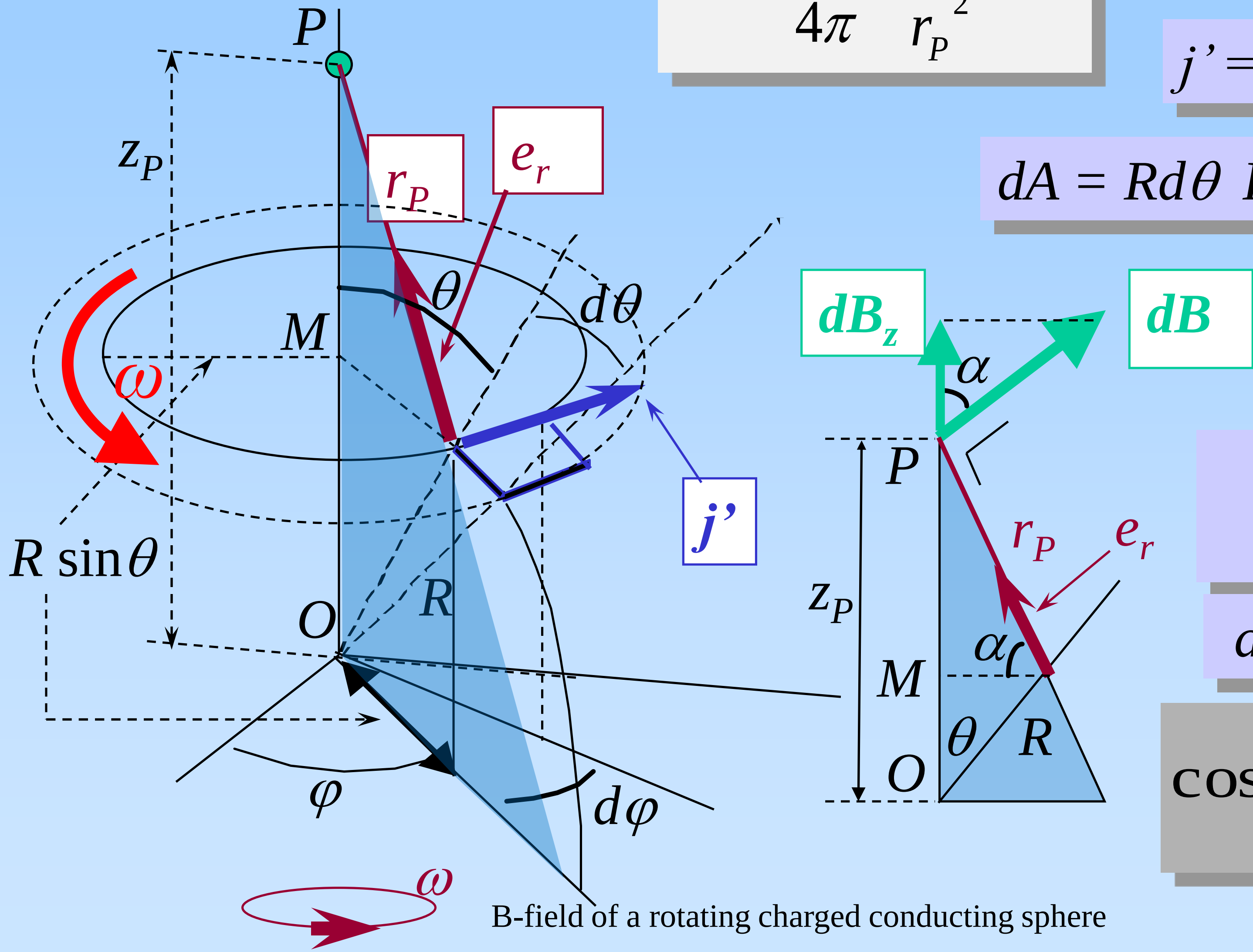
$dA = R d\theta\;\; R.\sin\theta.\, d\varphi$

Cylinder-symmetry:

Z-components only !!

$dB_z = dB.\cos\alpha$

$$\cos\alpha = \frac{R\sin\theta}{r_P}$$



Conducting sphere: on-axis (6)

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j}' \times \boldsymbol{e_r}}{r_P^{\,2}} dA$$

Needed:

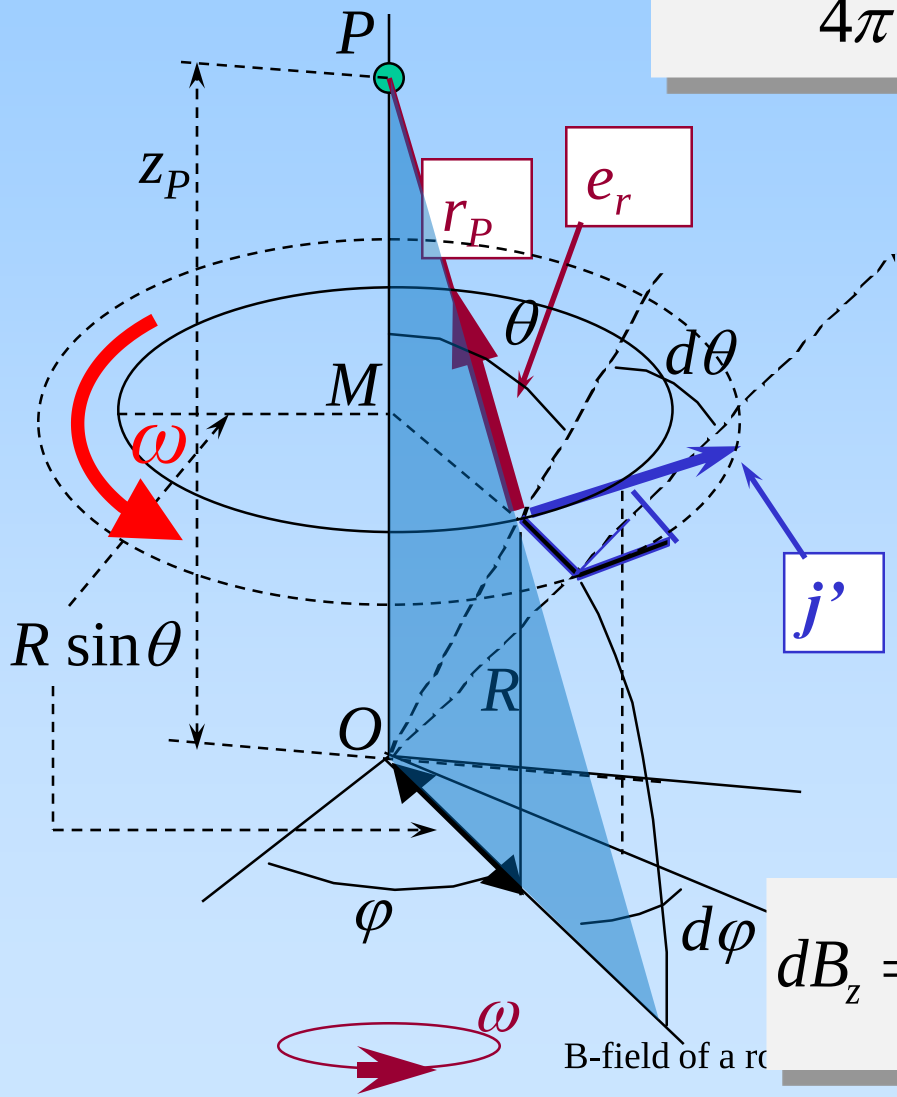
$dA,\ j\ ,\ \boldsymbol{e_r}\ ,\ r_P$

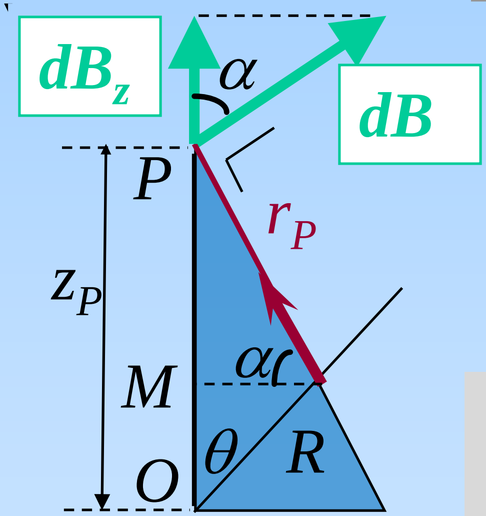
$$dA = Rd\theta .\ R.\sin\theta .\ d\varphi$$

$$j' = \sigma\omega R \sin\ \theta$$

$$\cos\alpha = \frac{R\sin\theta}{r_P}$$

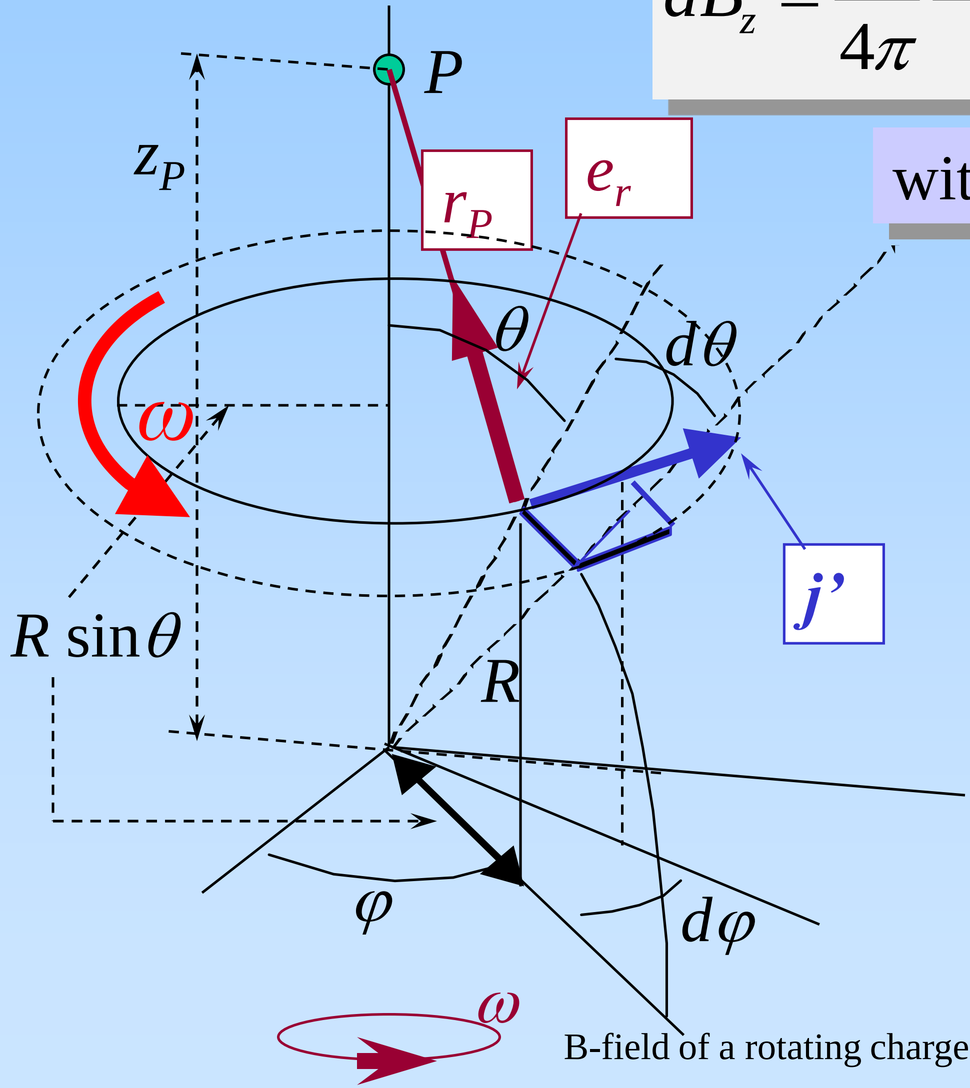
$$r_P^{\,2} = (R.\sin\theta)^2 + (z_P - R.\cos\theta)^2$$





$$dB_z = \frac{\mu_0}{4\pi}\frac{\sigma\omega R\sin\theta}{r_P^{\,2}} R.d\theta .R.\sin\theta d\varphi \frac{R\sin\theta}{r_P}$$

Conducting sphere: on-axis (7)



$$dB_z = \frac{\mu_0}{4\pi}\frac{\sigma\omega R\sin\theta}{r_P^{\;2}} R.d\theta.R.\sin\theta d\varphi \frac{R\sin\theta}{r_P}$$

with $r_P^{\,2} = (R.\sin\theta)^2 + (z_P - R.\cos\theta)^2$

$$dB_z = \frac{\mu_0}{4\pi}\frac{\sigma\omega R^4\sin^3\theta}{r_P^{\;3}} d\theta.d\varphi$$

Integration: $0 \leq \theta \leq \pi$

$0 \leq \varphi \leq 2\pi$

Conducting sphere: on-axis (8)

$$B_z = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \frac{\mu_0}{4\pi} \frac{\sigma\omega R^4 \sin^3\theta}{r_P{}^3}$$

with:

$r_P{}^2 = (R.\sin\theta)^2 + (z_P - R.\cos\theta)^2$

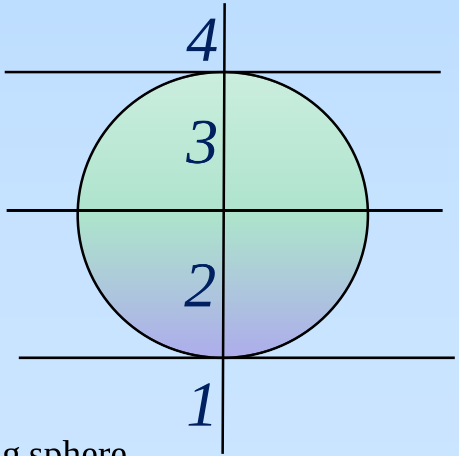
Set: $\frac{z_P}{R} = q$, and: $\cos\theta = x$, and with $a = 1+q^2$ and $b = -2q$:

$$B_z = \int_{-1}^{+1} \frac{x^2-1}{(a+bx)^{3/2}} dx = \frac{8}{3b^3}\left[(b-2a)\sqrt{a+b} + (b+2a)\sqrt{a-b}\right]$$

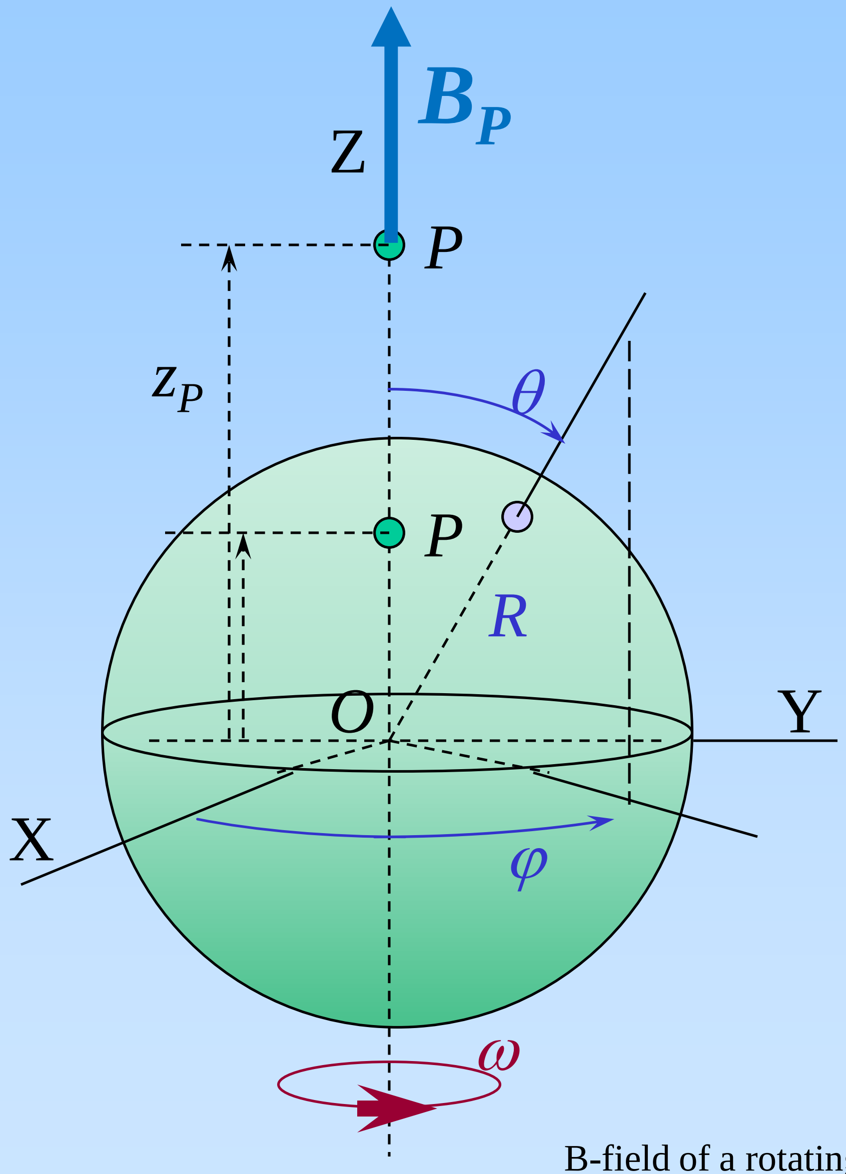
(Set $a+bx = y$, and express dx and x^2-1 in dy and y, and integrate…)

4 solutions, depending on √(..)-terms:

1. $z_P \leq -R$
2. $-R \leq z_P \leq 0$
3. $0 \leq z_P \leq R$
4. $z_P \geq R$



Conducting sphere: on-axis (9)



$$\text{result : } \mathbf{B}_P = \frac{2\mu_0 \sigma \omega R^4}{3.z_p^{\ 3}} \mathbf{e_z}$$

this result holds for $z_P > R$;

for -R<z_P<R the result is:

$$\mathbf{B}_P = \frac{2}{3} \mu_0 \sigma \omega R \, \mathbf{e_z}$$

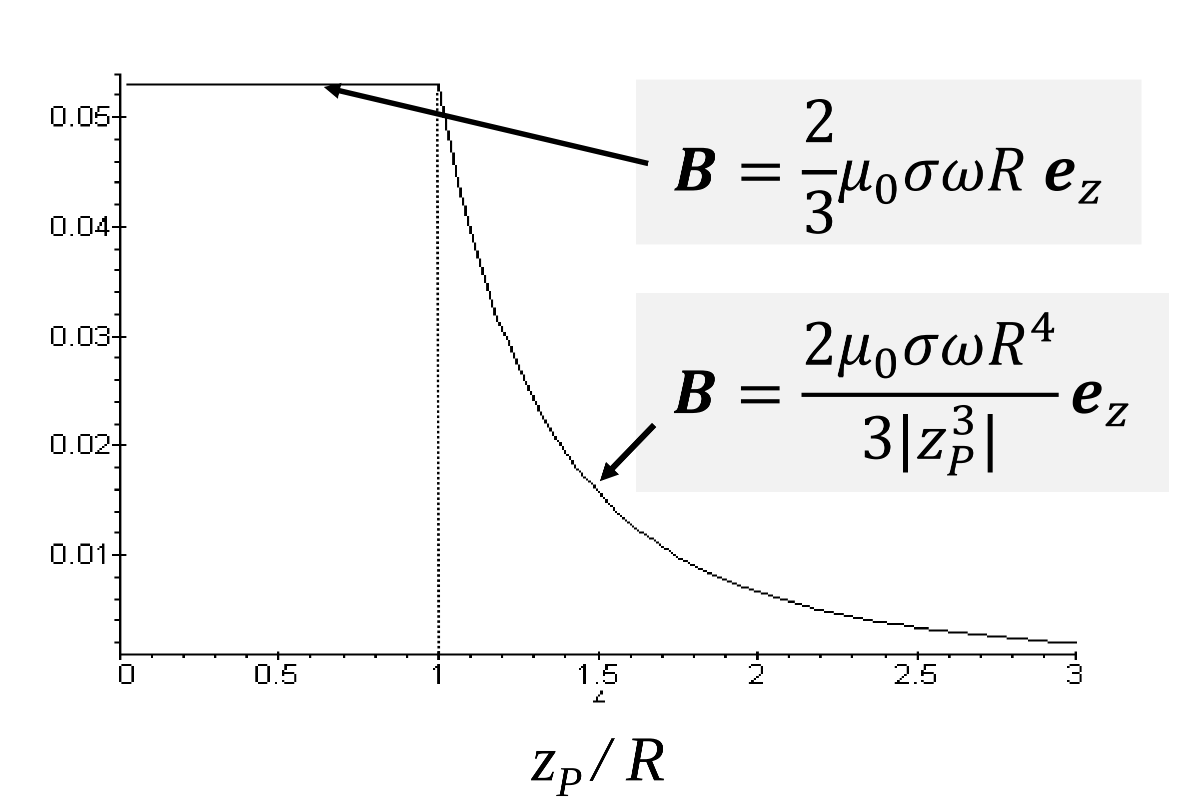
and for z_P<-R:

$$\mathbf{B}_P = \frac{2\mu_0 \sigma \omega R^4}{-3.z_p^{\ 3}} \mathbf{e_z}$$

B directed along +$\boldsymbol{e}_{\mathbf{z}}$ for all points everywhere on Z-axis !!

inside sphere: constant field !!

Conducting sphere: on-axis (10)



$$\boldsymbol{B} = \frac{2}{3}\mu_0 \sigma \omega R \, \boldsymbol{e}_z$$

$$\boldsymbol{B} = \frac{2\mu_0 \sigma \omega R^4}{3|z_P^3|} \boldsymbol{e}_z$$

Plot of ***B*** for:

$Q = 1$

$\mu_0 = 1$

$\omega = 1$

$R = 1$

(in SI-units)

$Q = \sigma . 4\pi R^2$

Conclusion: inside conducting sphere: **on-axis**: field = constant.

Question: what about the field inside the sphere, but **off-axis?**

To be investigated in part II === >

Conclusions for on-axis (1)

Conducting sphere

$|z_P| > R$

$$\boldsymbol{B} = \frac{\mu_0 Q \omega R^2}{6\pi \left| z_p{}^3 \right|} \boldsymbol{e_z}$$

$|z_P| < R$

$$\boldsymbol{B} = \frac{\mu_0 Q \omega}{6\pi R} \boldsymbol{e_z}$$

$$Q = \sigma . 4\pi R^2$$

Homogeneously charged sphere

(see other presentation)

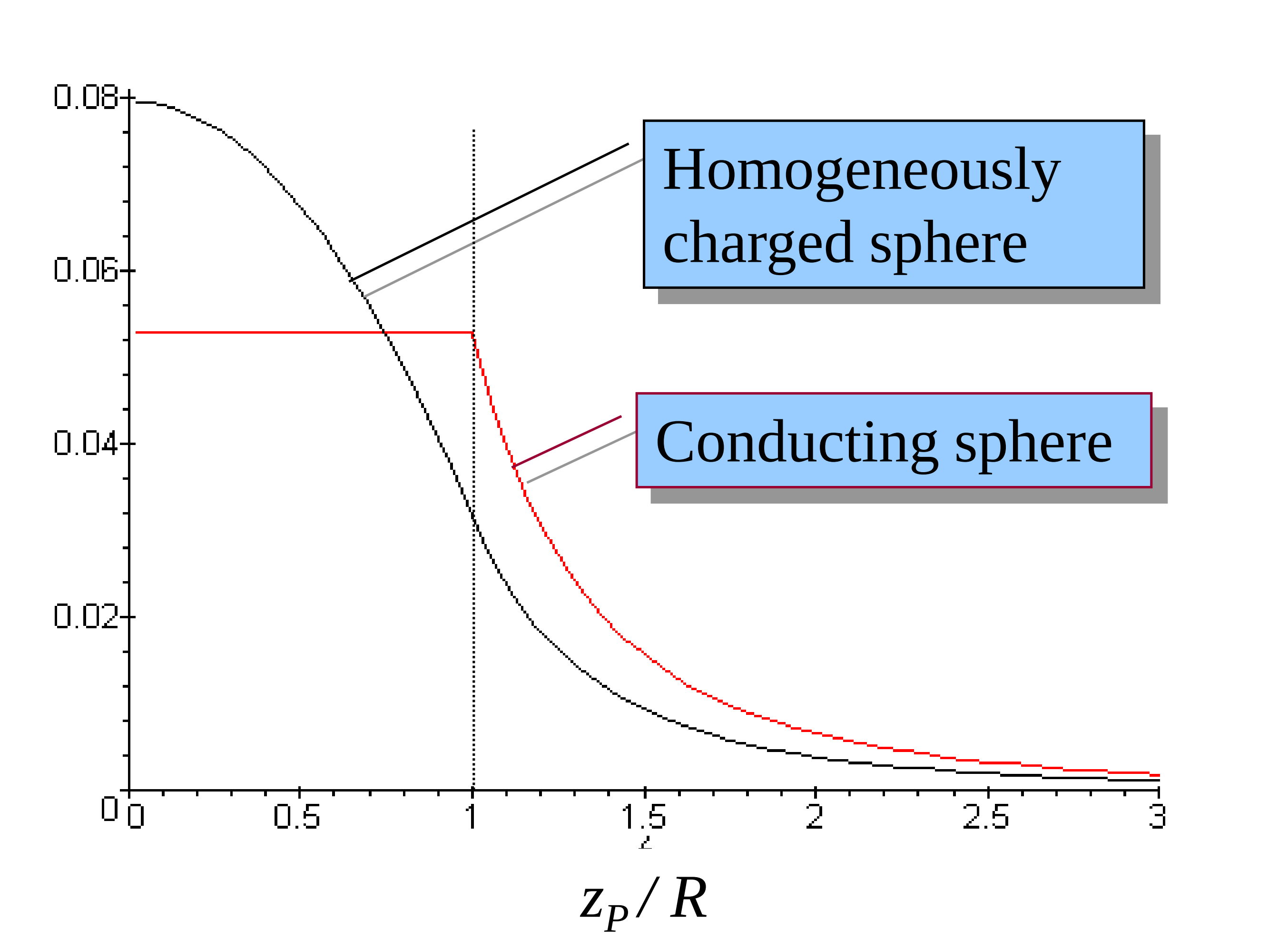
$|z_P| > R$

$$\boldsymbol{B} = \frac{\mu_0 Q \omega R^2}{10\pi \left| z_P{}^3 \right|} \boldsymbol{e_z}$$

$|z_P| < R$

$$\boldsymbol{B} = \frac{\mu_0 Q \omega}{20\pi R^3} \left(5R^2 - 3 z_P{}^2 \right) \boldsymbol{e_z}$$

Conclusions for on-axis (2)



Plot of ***B*** for:

$Q = 1$

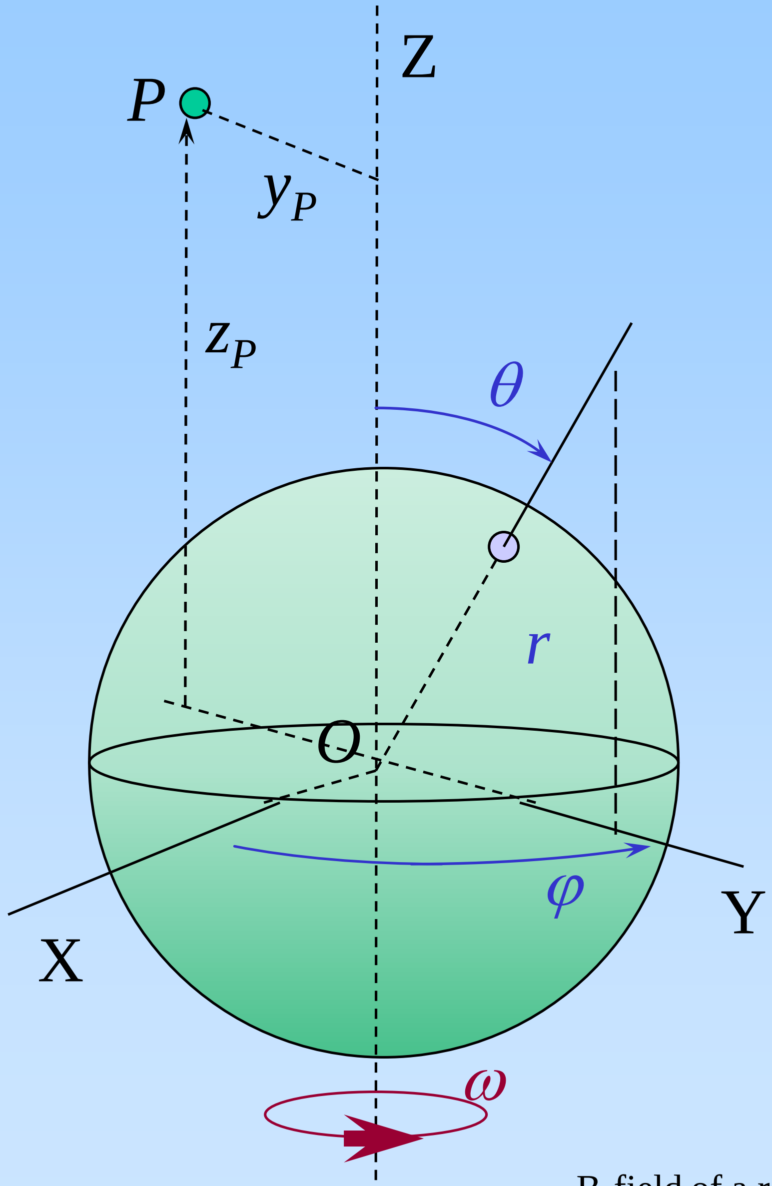
$\mu_0 = 1$

$\omega = 1$

$R = 1$

(in SI-units)

Off-axis: Analysis and Symmetry (1)



Part II. Calculate ***B***-field in point P

off *the axis of rotation* (Z-axis)

inside or outside the sphere

Rotation axis (Z-axis) =

= symmetry axis .

Assume P (0, y_P , z_P) in YZ-plane.

Coordinate systems:

- X,Y, Z
- r, θ, φ

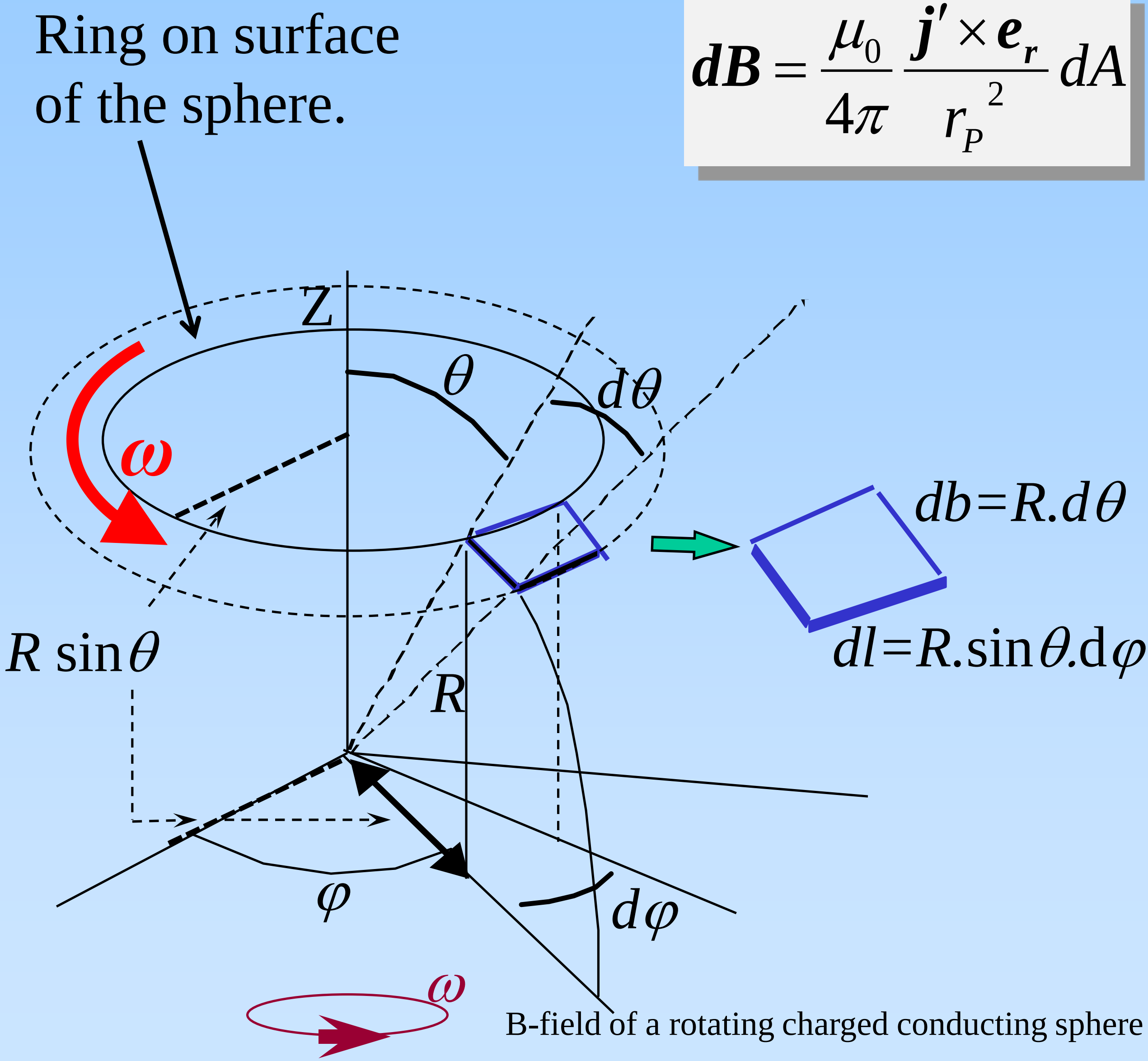
Conducting sphere: off-axis (1)

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j}'\times\boldsymbol{e_r}}{r_P^{\ 2}}\,dA$$

with $\boldsymbol{j}'$ in [A/m]

Surface charge $\sigma.dA$ on dA will rotate with ω

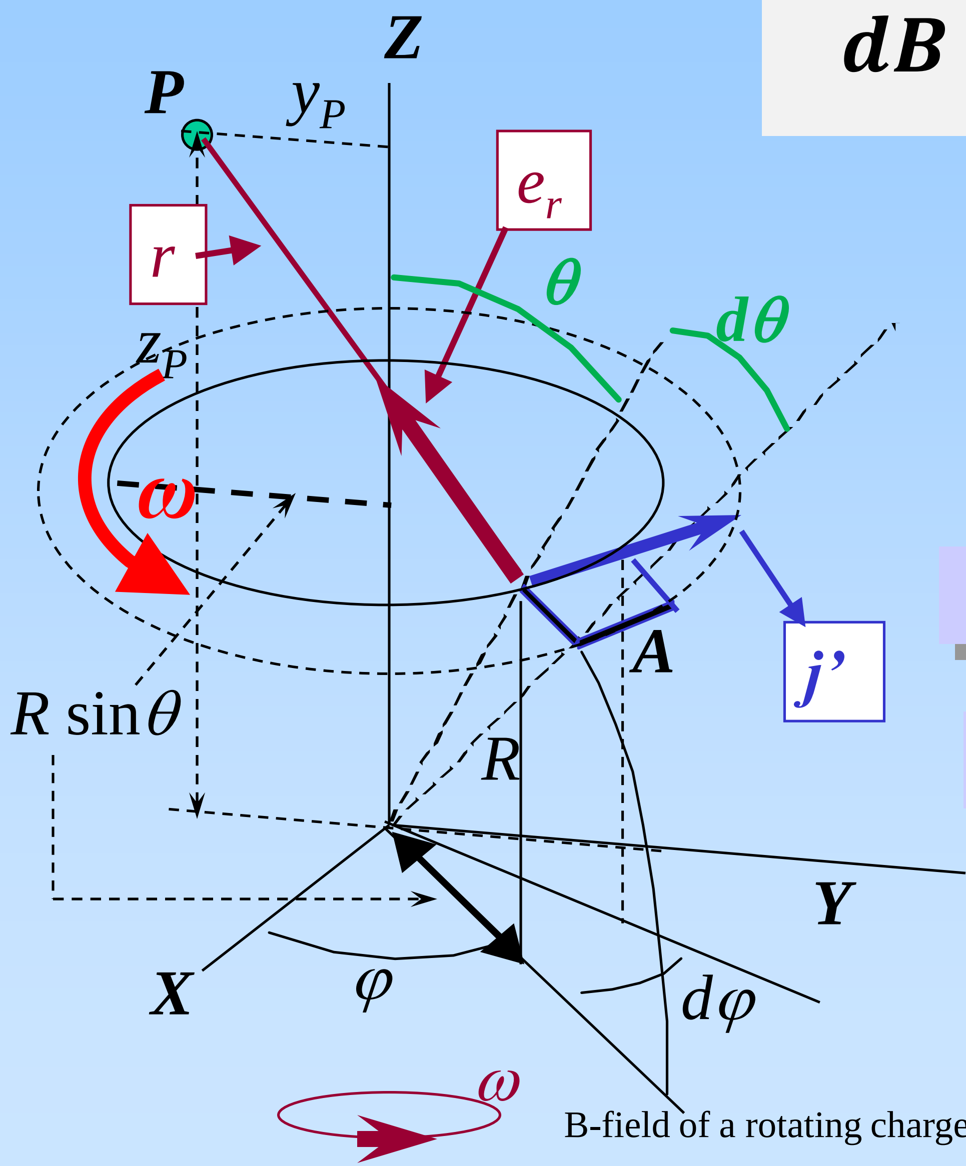
dA = width db • length dl



Needed:

- $\boldsymbol{j}'$, $\boldsymbol{e_r}$, r_P

Conducting sphere: off-axis (2)



$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j'}\times\boldsymbol{e_r}}{\boldsymbol{r}^2}\,dA = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j'}\times\boldsymbol{r}}{\boldsymbol{r}^3}\,dA.$$

$$dA = R.d\theta.\;\; R.\sin\theta.\,d\varphi$$

$$j' = \sigma\omega R \sin\,\theta$$

Off-axis: $\boldsymbol{j'}$ ***not*** $\perp$ $\boldsymbol{e_r}$!!

$$\boldsymbol{j'} = j'(-\sin\varphi.\boldsymbol{e_x} + \cos\varphi.\boldsymbol{e_y} + 0.\boldsymbol{e_z})$$

$\boldsymbol{r} = \boldsymbol{r_P} - \boldsymbol{r_A}$; **A** = at dA-element

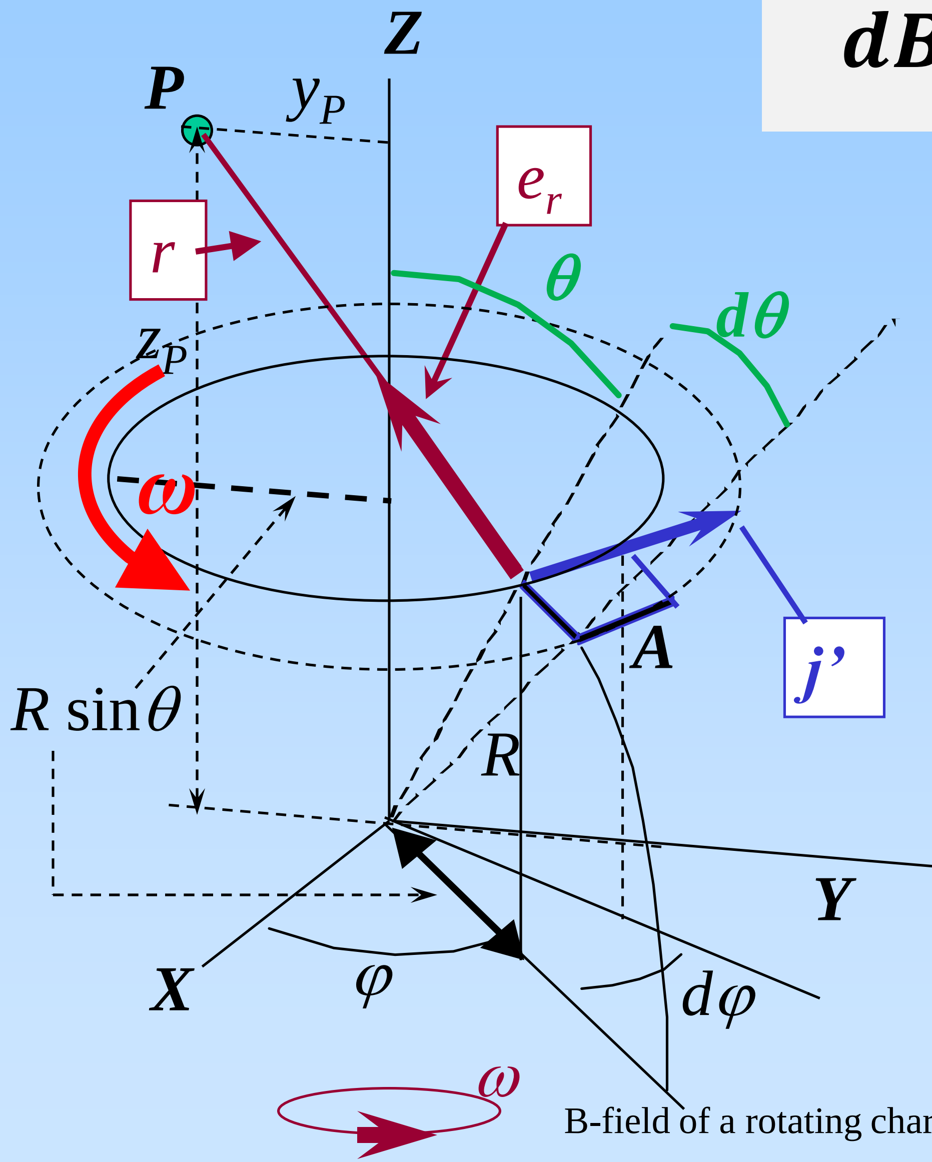
$$\boldsymbol{r_P} = (0.\boldsymbol{e_x} + y_P.\boldsymbol{e_y} + z_P.\boldsymbol{e_z})$$

$$\boldsymbol{r_A} = R\,(\sin\theta.\sin\varphi.\boldsymbol{e_x} + \sin\theta.\cos\varphi.\boldsymbol{e_y} + \cos\theta.\boldsymbol{e_z})$$

Conducting sphere: off-axis (2)

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j'}\times\boldsymbol{e_r}}{\boldsymbol{r^2}}dA = \frac{\mu_0}{4\pi}\frac{\boldsymbol{j'}\times\boldsymbol{r}}{\boldsymbol{r^3}}dA.$$

$$dA = R.d\theta.\ R.\sin\theta.\ d\varphi$$



B-field of a rotating charged conductin…

θ

!!

z)

nt

z)

Analytical approach: not feasible

(due to $\boldsymbol{j} \times \boldsymbol{r}$ and r^3)

Numerical approach necessary.

$$\boldsymbol{r_A} = R\,(\sin\theta.\sin\varphi.\boldsymbol{e_x} + \sin\theta.\cos\varphi.\boldsymbol{e_y} + \cos\theta.\boldsymbol{e_z})$$

Conducting sphere: off-axis (4)

Available for download on www.demul.net/frits:
offline program: EM_solenoids
in file: EM_programs.zzz
on subpage Electromagnetism

This program can calculate:
B*-** and ***A-fields for:

- Single solenoids
- Pairs of solenoids
- Dipole fields
- Field of a rotating charged conducting sphere
- and sphere segments

Examples

Examples

1. Rotating charged conducting sphere

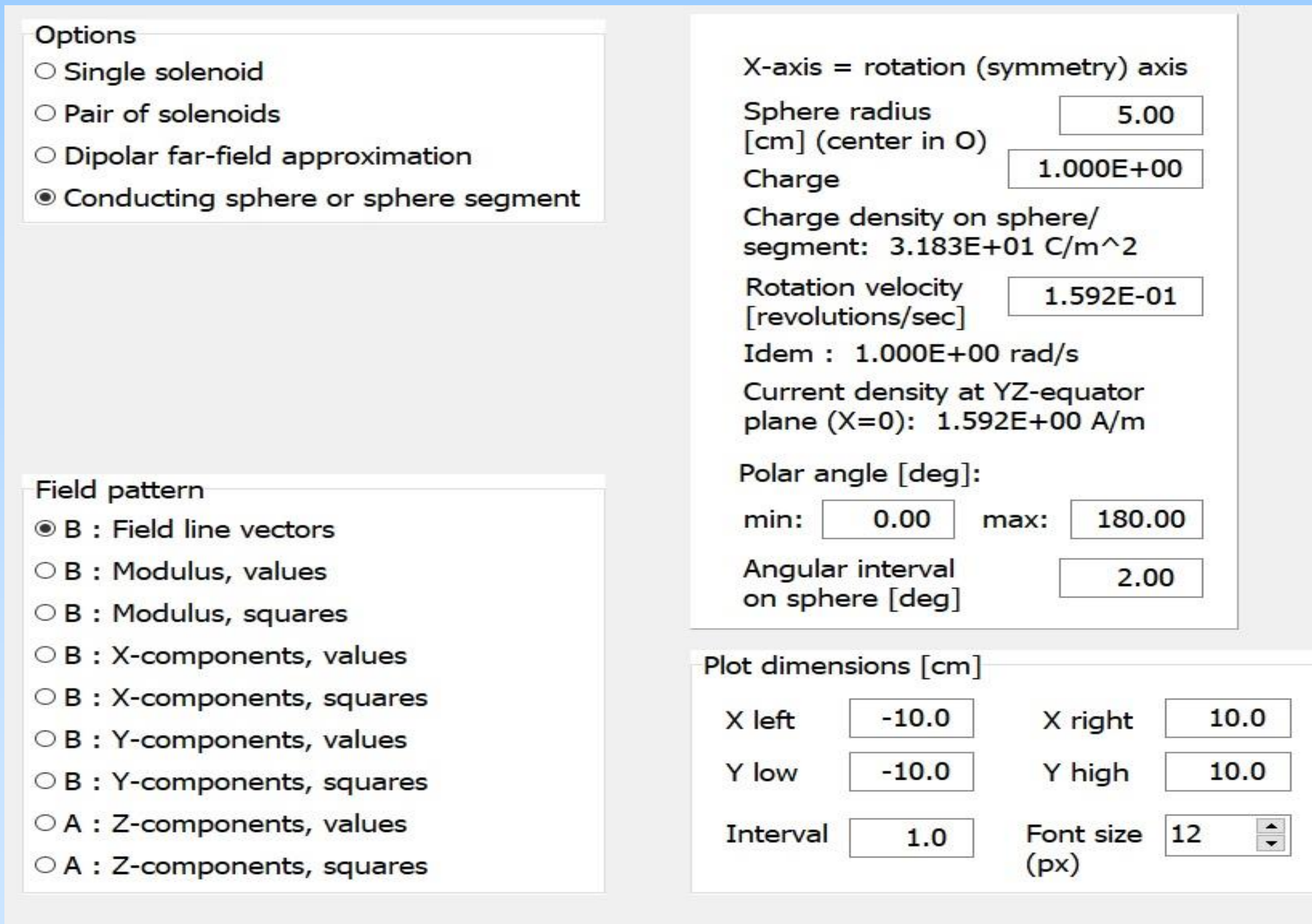
Properties:

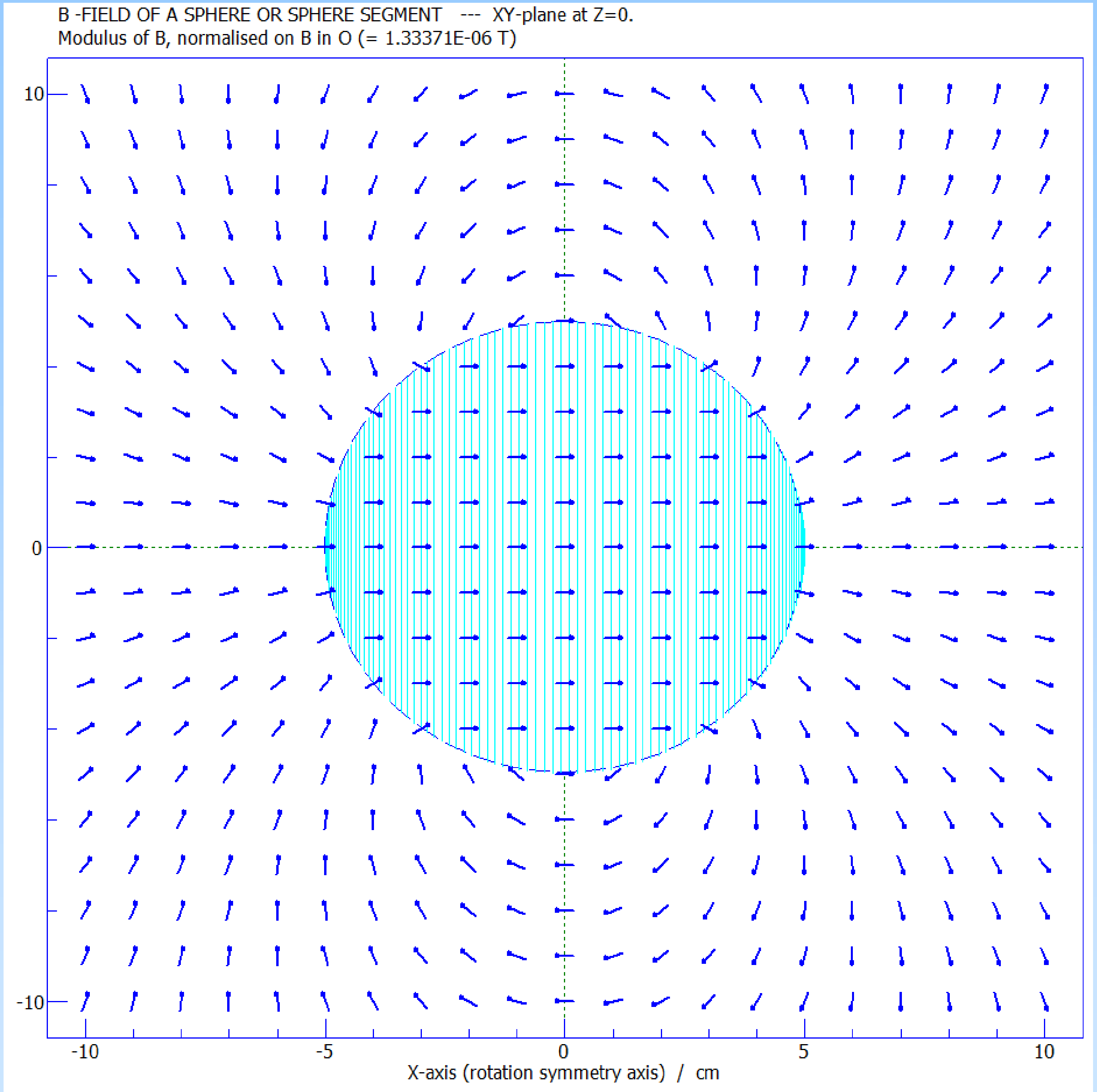
- Charge = 1 C
- Radius = 5 cm
- Velocity = 1 rad/s = 0.1592 rev./s

NB. Rotation axis = symmetry axis = X-axis;
Fields shown in XY-plane at Z=0.

Examples

1. Rotating charged conducting sphere: settings:

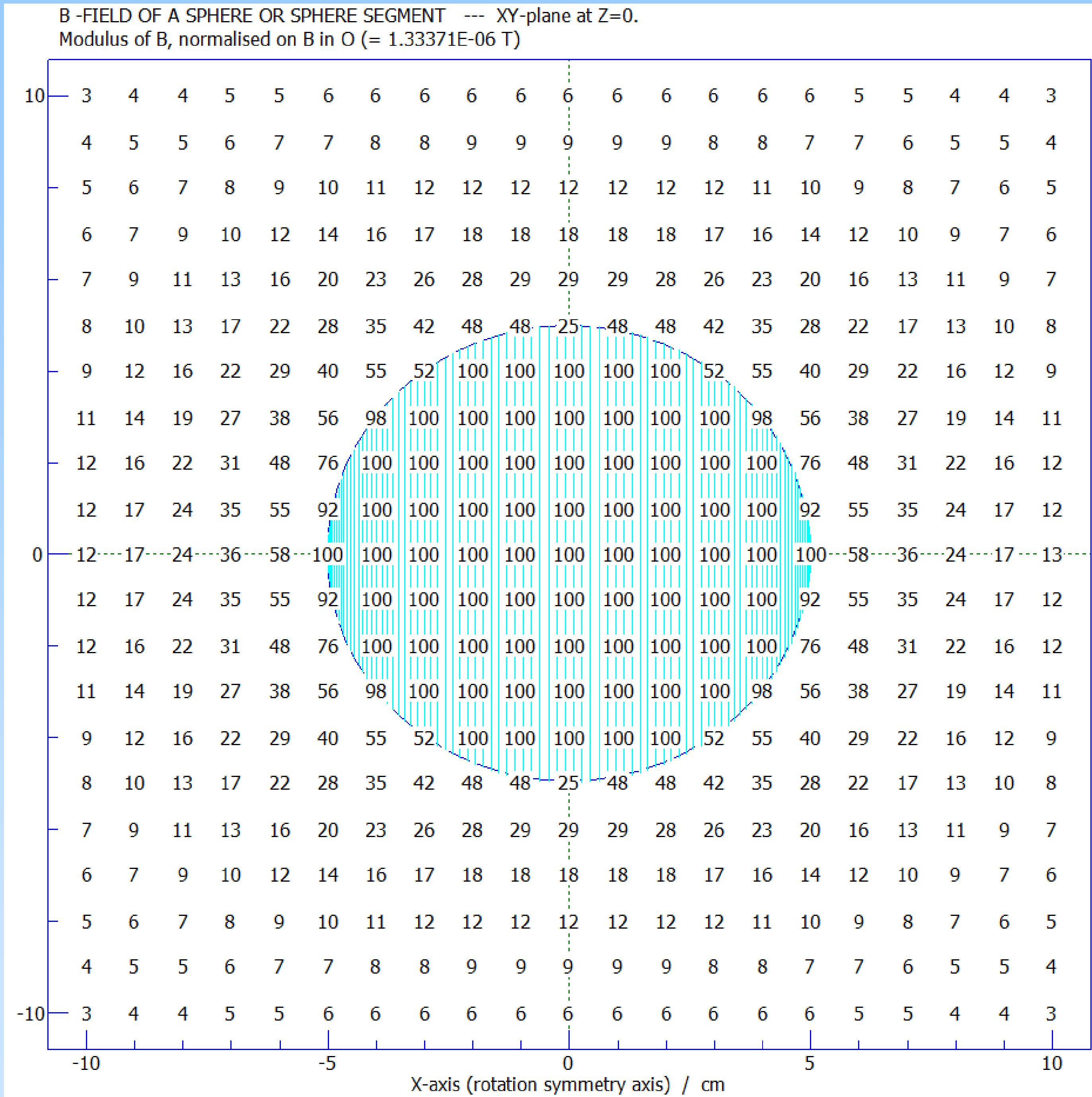




B-field: Sphere rotating around X-axis

Inside the sphere: homogeneous field

Field strength inside = 1.3337 μT



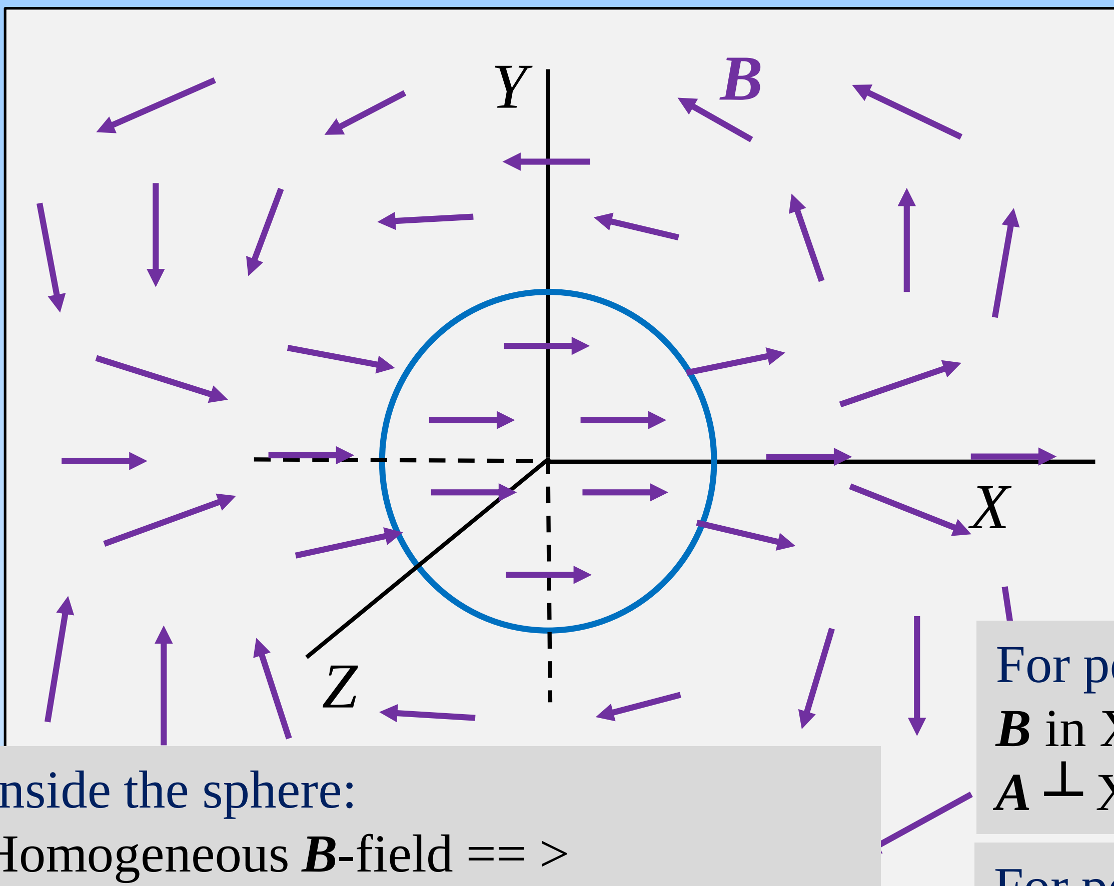
B-field: Sphere rotating around X-axis

Inside the sphere: homogeneous field

Field strength inside = 1.3337 μT

Conducting sphere rotating around X-axis : *B* and *A*-fields

B-field: Cross section of sphere: XY-plane at Z=0:



Expression for a surface current:

$$\boldsymbol{dB} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{j'} \times \boldsymbol{e_r}}{r^2} dA$$

***A*-field**: **Vector potential**:
B = rot ***A*** (= curl ***A***)

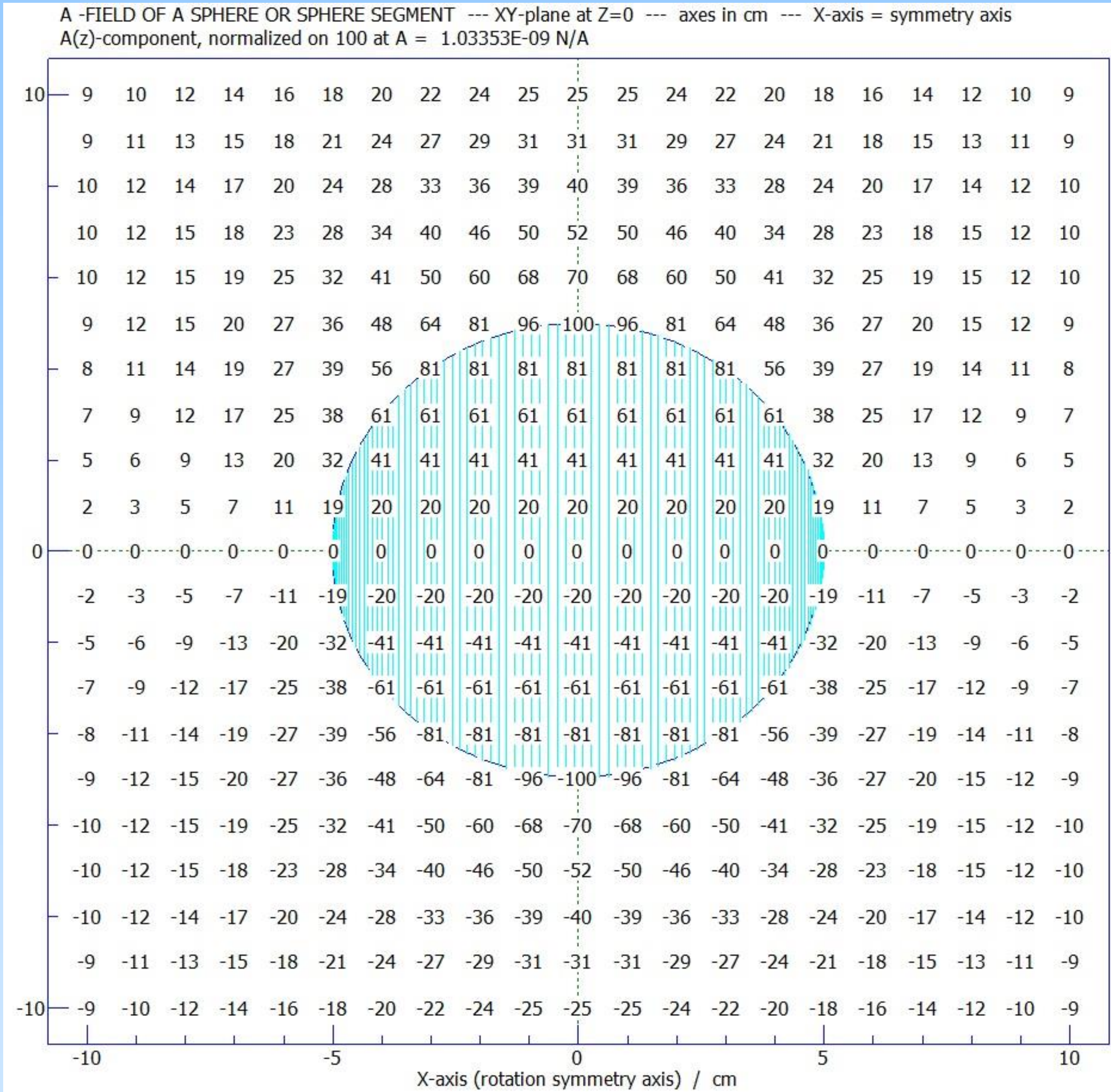
$$\boldsymbol{dA} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{j'}}{r} dA$$

B and ***A*** :
perpendicular fields.

For points in XY-plane:
B in XY-plane, no Z-component
A ⊥ XY-plane, Z-component only.

For points outside XY-plane:
Cylindrical symmetry around X-axis.

Inside the sphere:
Homogeneous ***B***-field == >
A-field varies linearly with y-coordinate
(due to derivatives in *rot* (*curl*))



A-field:
Sphere rotating around X-axis

A = 0 at rotation symmetry axis

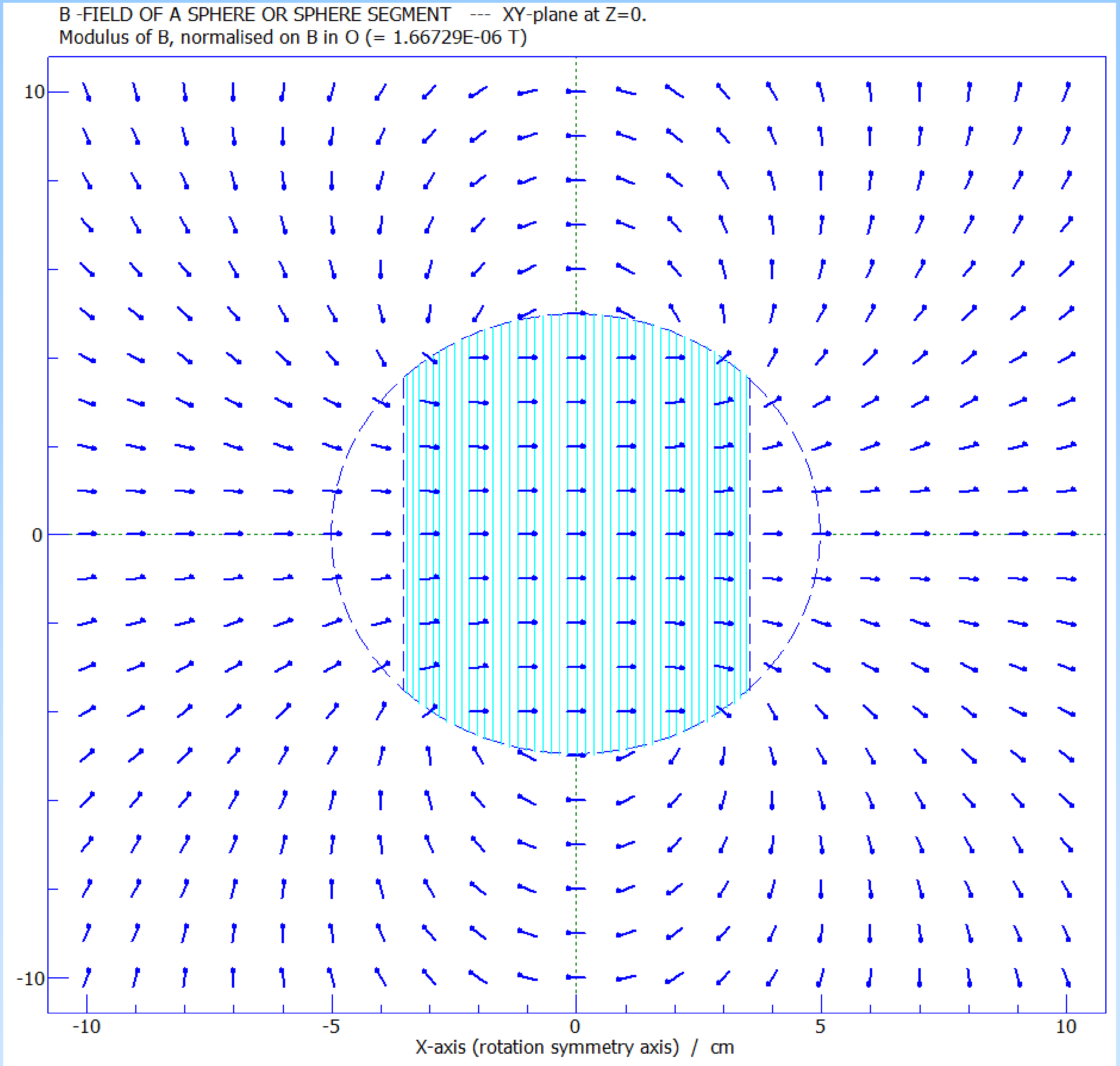
Examples

Examples

2. Rotating charged conducting sphere segment between 45^0 and 135^0 (ring shape)

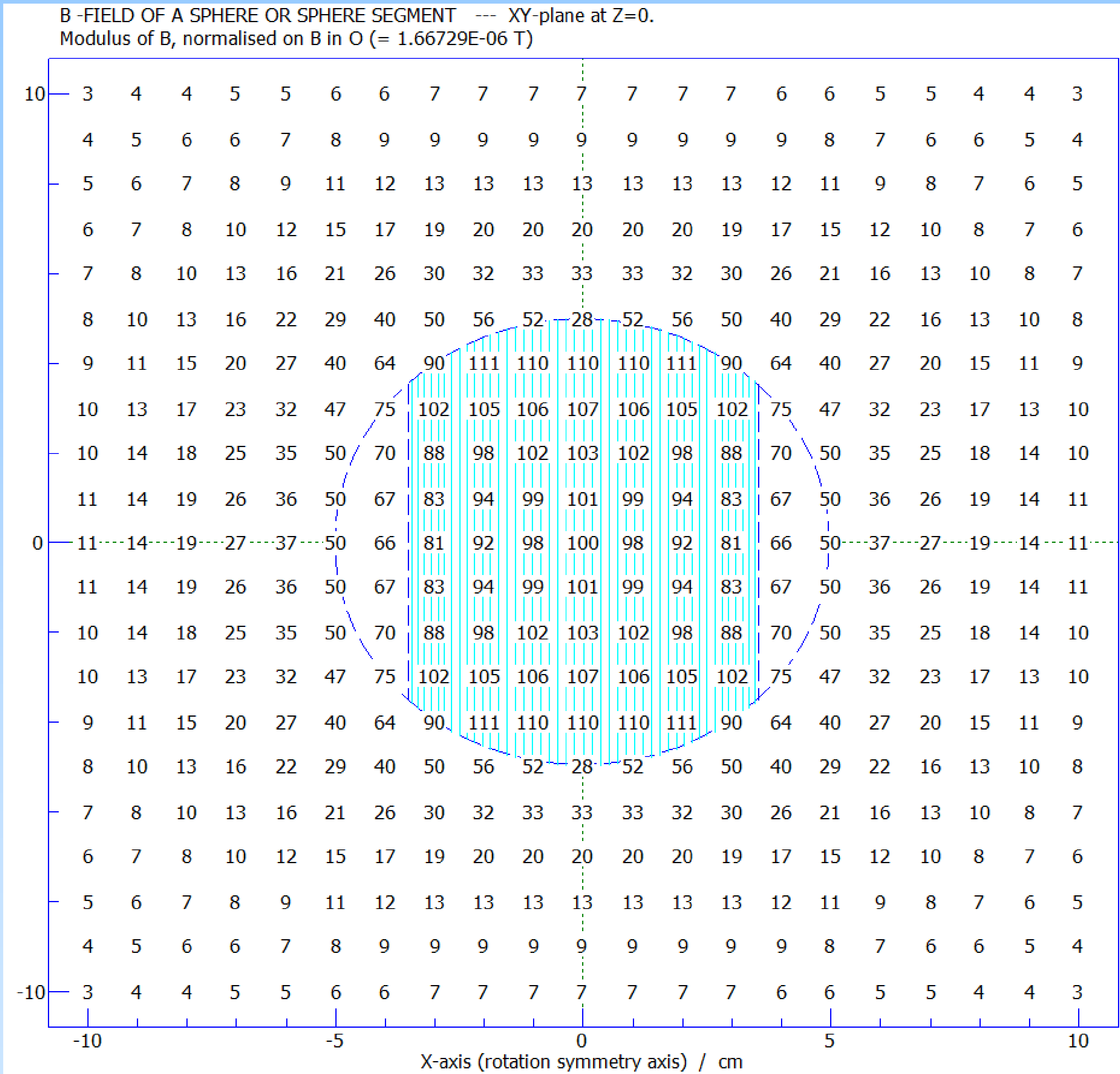
Properties:

- Charge = 1 C
- Radius = 5 cm
- Velocity = 1 rad/s = 0.1592 rev./s



B-field:
Sphere segment (ring shape) rotating around X-axis

Field already looks like a solenoid field



B-field: Sphere segment (ring shape) rotating around X-axis

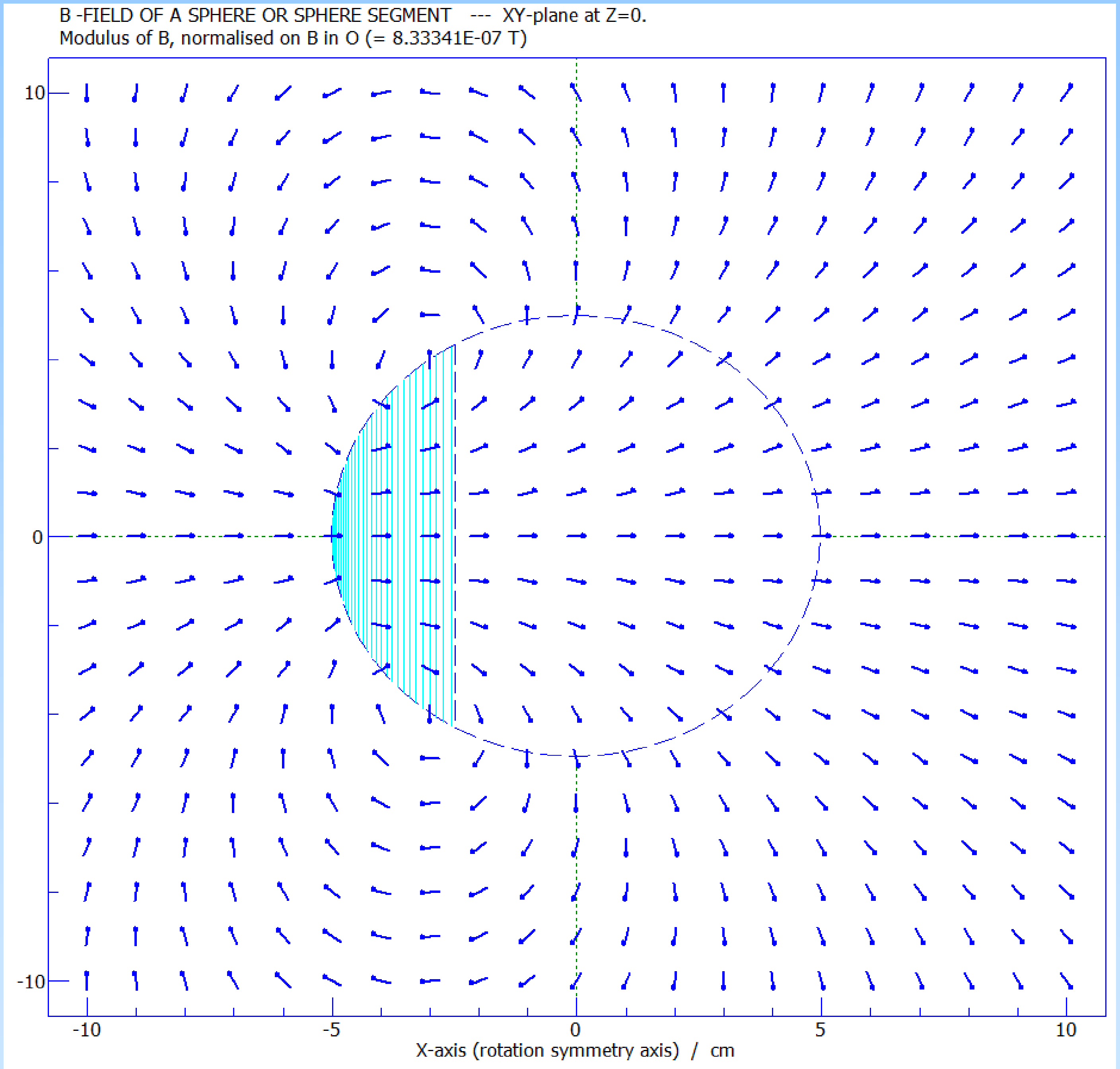
Field already looks like a solenoid field

Examples

3. Rotating charged conducting sphere segment between 120^0 and 180^0 (bowl shape)

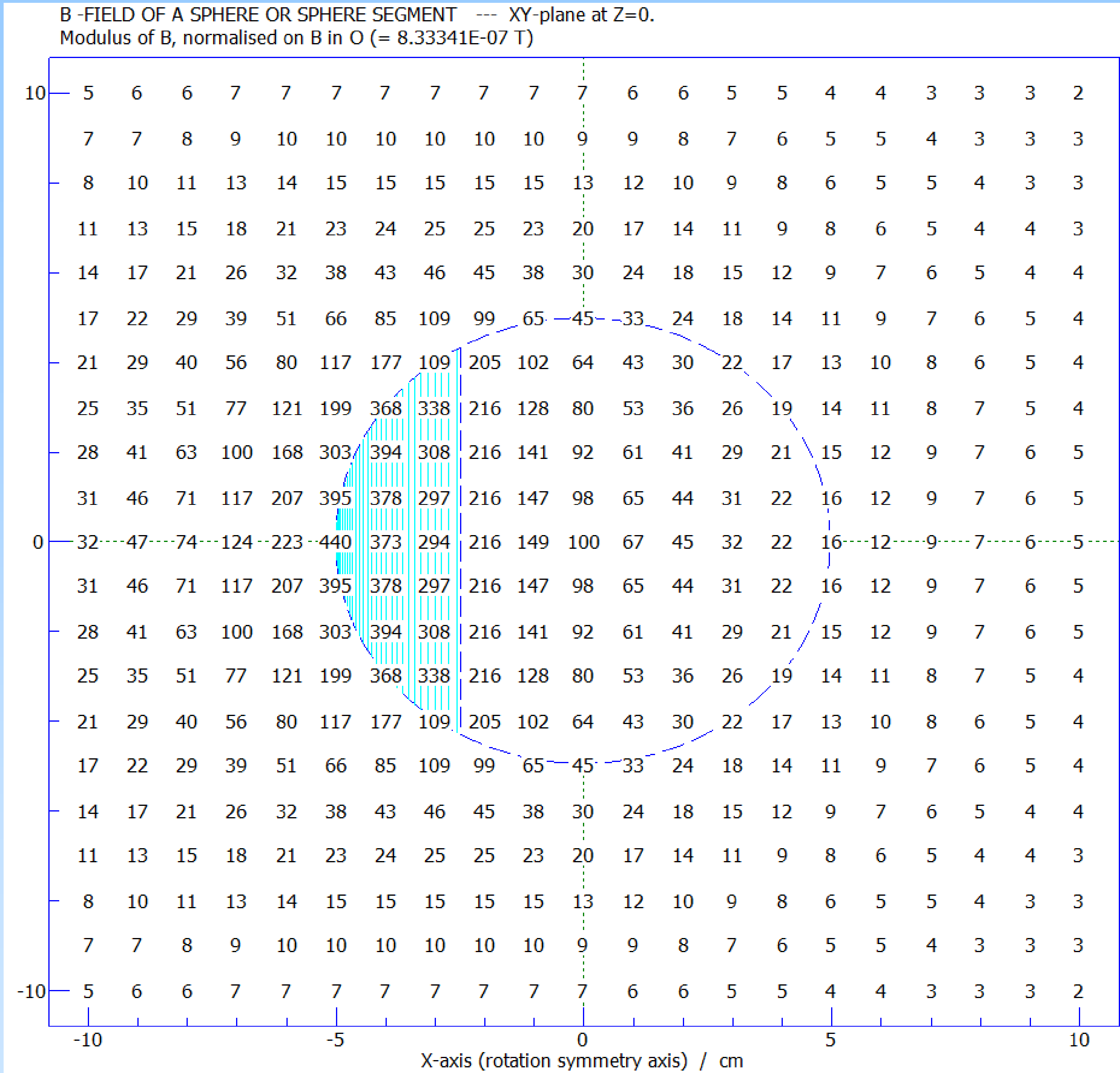
Properties:

- Charge = 1 C
- Radius = 5 cm
- Velocity = 1 rad/s = 0.1592 rev./s



B-field: Sphere segment (bowl shape) rotating around X-axis

Field already looks like a dipolar field



B-field:
Sphere segment (bowl shape) rotating around X-axis

Field already looks like a dipolar field

the end