Magnetic Field of a Rotating Charged Conducting Sphere

2nd version: on-axis and off-axis

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Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss’ Law for a cylindrical charge
- Gauss’ Law for a charged plane
- Laplace’s and Poisson’s Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

Presentations and programs (free) can be downloaded from: www.demul.net/frits
Available:

A charged conducting sphere (charge $Q$, radius $R$), rotating with $\omega$ rad/sec

Question:

Calculate $B$-field in arbitrary points inside and outside the sphere

I. on the axis of rotation

II. off-axis

Ad. I : analytical approach possible

Ad. II : numerical approach needed
Objective:

**B-field:** of a charged conductive sphere rotating around the X-axis.

Inside the sphere:

homogeneous field
Analysis and Symmetry for on-axis (1)

Assume P on Z-axis.

**Part I.** Calculate \( B \)-field in point \( P \) on the axis of rotation (Z-axis) inside or outside the sphere

(\textbf{Part II : points P off-axis } )

Coordinate systems:
- \( X,Y,Z \)
- \( r, \theta, \varphi \)

Symmetry: around rotation axis.
Analysis and Symmetry for on-axis (2)

Conducting sphere,
all charges at surface:
surface charge density:
\[ \sigma = \frac{Q}{4\pi R^2} \text{ [C/m}^2] \]

Rotating charges will establish a “surface current”,
directed along surface circles.

Surface current density \( j' \) [A/m]:
will be a function of \( \theta \)

B-field of a rotating charged conducting sphere.
B-field of a rotating charged conducting sphere

**Analysis and Symmetry for on-axis (3)**

Cylindrical symmetry around Z-axis:
- \( dB \perp dl \) and \( e_r \).
- If \( P = \) on-axis: \( dl \perp e_r \)

**Direction of **\( dB **:**

- **Z-comp. only !!**
- **X- and Y-comp. cancel.**

**Biot & Savart:**

\[
\frac{dB}{4\pi} = \frac{\mu_0 I \cdot dl \times e_r}{r_P^2}
\]
Biot & Savart:

$$dB = \frac{\mu_0}{4\pi} \frac{I. dl \times e_r}{r_p^2}$$

$dB$, $dl$ and $e_r$ mutually perpendicular

Question:
How to relate $I. dl$ (in A.m) to surface current density $j$ (in A/m$^2$)
Intermezzo: a surface current

Biot & Savart:

\[ dB = \frac{\mu_0}{4\pi} \frac{I.dl \times e_r}{r_p^2} \]

Current strip at surface:
\( j' \): current density [A/m]

NB. Density [A/m] = current per m width!

\[ I.dl = j'.db.dl = j'.db.dl = j'.dA \]

Biot & Savart:

\[ dB = \frac{\mu_0}{4\pi} \frac{j' db.dl \times e_r}{r_p^2} = \frac{\mu_0}{4\pi} \frac{j' \times e_r}{r_p^2} dA \]

\( dB \perp dl \) and \( e_r \)

B-field of a rotating charged conducting sphere
Analysis and Symmetry for on-axis (4)

Biot & Savart:

\[ dB = \frac{\mu_0}{4\pi} \frac{j'db.dl \times e_r}{r_p^2} = \frac{\mu_0}{4\pi} \frac{j' \times e_r}{r_p^2} dA \]

Required:
expressions for:
\(dA\), \(j'\), \(e_r\), \(r_p\)

\(dB\), \(dl\) and \(e_r\)
mutually perpendicular
Conducting sphere: on-axis (1)

**Ring** on surface of the sphere.

Conducting sphere, surface density:
\[ \sigma = \frac{Q}{4\pi R^2} \]

Surface element:
\[ dA = (R \cdot d\theta) \cdot (R \cdot \sin \theta \cdot d\phi) \]

Needed: expressions for:
- \( dA \)
- \( j' \)
- \( e_r \)
- \( r_P \)

Projection of **ring** on XY-plane, radius = \( R \sin \theta \)

Projection of ring on XY-plane, radius = \( R \sin \theta \)
Conducting sphere: on-axis (2)

Ring on surface of the sphere.

\[ dB = \frac{\mu_0}{4\pi} \frac{j' \times e_r}{r_p^2} dA \]

Needed: expressions for: \( dA, j', e_r, r_P \)

with \( j' \) in [A/m]

Surface charge \( \sigma.dA \) on \( dA \) will rotate with \( \omega \)

\[ dA = db \cdot dl \]
Conducting sphere: on-axis (3)

\[ \text{Ring on surface: area} = 2\pi (R \sin \theta) (Rd\theta) \]

\[ \text{Charge on ring:} \quad \sigma \cdot 2\pi R \sin \theta \cdot Rd\theta \]

\[ \text{Full rotation over} \ 2\pi \ \text{in} \ 2\pi/\omega \ \text{s.} \]

\[ \text{current:} \quad dI = \sigma \cdot 2\pi R \sin \theta \cdot Rd\theta / (2\pi/\omega) \]

\[ = \sigma \omega R \sin \theta \cdot Rd\theta \]

\[ \text{current density:} \quad j' = dI / (Rd\theta) = \sigma \omega R \sin \theta \quad [\text{A/m}] \]

\[ \text{Needed:} \quad dA, j', \sigma, r_P \]

B-field of a rotating conductor

\[ R.d\theta \quad R.\sin \theta. d\theta \]

\[ \theta \quad d\theta \]

\[ \omega \]

\[ Z \]

\[ \omega \]

\[ \theta \]

\[ d\theta \]

\[ \varphi \]

\[ d\varphi \]
Conducting sphere: on-axis (4)

\[ dB = \frac{\mu_0}{4\pi} \frac{j' \times e_r}{r_P^2} dA \]

Needed:
\(dA, j', e_r, r_P\)

\[ dA = R.\sin\theta. d\phi. R. d\theta. \]

\[ j' = \sigma \omega R \sin \theta \]

\[ j' \perp e_r : \]
\[ \Rightarrow |j' \times e_r| = j'.|e_r| = j' \]

B-field of a rotating charged conducting sphere
Conducting sphere: on-axis (5)

\[ dB = \frac{\mu_0}{4\pi} \frac{j' \times e_r}{r_P^2} dA \]

\[ j' = \sigma \omega R \sin \theta \]

\[ dA = Rd\theta \ R.\sin\theta. \ d\phi \]

\[ dB_z = dB \cdot \cos \alpha \]

\[ \cos \alpha = \frac{R \sin \theta}{r_P} \]

B-field of a rotating charged conducting sphere
\[ dB = \frac{\mu_0}{4\pi} \frac{j' \times e_r}{r^2_P} dA \]

\[ dA = Rd\theta \cdot R \cdot \sin \theta \cdot d\phi \]

\[ j' = \sigma \omega R \sin \theta \]

\[ \cos \alpha = \frac{R \sin \theta}{r_P} \]

\[ r^2_P = (R \sin \theta)^2 + (z_P - R \cos \theta)^2 \]

\[ dB_z = \frac{\mu_0}{4\pi} \frac{\sigma \omega R \sin \theta}{r^2_P} R d\theta \cdot R \sin \theta d\phi \frac{R \sin \theta}{r_P} \]

Needed:
- \( dA \), \( j \), \( e_r \), \( r_P \)
Conducting sphere: on-axis (7)

\[ dB_z = \frac{\mu_0}{4\pi} \frac{\sigma \omega R \sin \theta}{r_p^2} R d\theta R \sin \theta d\varphi \frac{R \sin \theta}{r_p} \]

with \( r_p^2 = (R \sin \theta)^2 + (z_p - R \cos \theta)^2 \)

\[ dB_z = \frac{\mu_0}{4\pi} \frac{\sigma \omega R^4 \sin^3 \theta}{r_p^3} d\theta d\varphi \]

Integration:

\[ 0 \leq \theta \leq \pi \]
\[ 0 \leq \varphi \leq 2\pi \]
Conducting sphere: on-axis (8)

\[ B_z = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \frac{\mu_0 \sigma \omega R^4 \sin^3 \theta}{4\pi r_P^3} \]

with:
\[ r_P^2 = (R \sin \theta)^2 + (z_P - R \cos \theta)^2 \]

Set: \[ \frac{z_P}{R} = q \), and: \( \cos \theta = x \), and with \( a = 1 + q^2 \) and \( b = -2q \):

\[ B_z = \int_{-1}^{+1} \frac{x^2 - 1}{(a + bx)^{3/2}} \, dx = \frac{8}{3b^3} \left[ (b - 2a)\sqrt{a + b} + (b + 2a)\sqrt{a - b} \right] \]

(Set \( a + bx = y \), and express \( dx \) and \( x^2 - 1 \) in \( dy \) and \( y \), and integrate…)

4 solutions, depending on \( \sqrt{\text{(..)}} \)-terms:

1. \( z_P \leq -R \)
2. \( -R \leq z_P \leq 0 \)
3. \( 0 \leq z_P \leq R \)
4. \( z_P \geq R \)

B-field of a rotating charged conducting sphere
Conducting sphere: on-axis (9)

result: \[ \mathbf{B}_P = \frac{2 \mu_0 \sigma \omega R^4}{3.z_p^3} \mathbf{e}_z \]

this result holds for \( z_p > R \);

for \(-R < z_p < R\) the result is:

\[ \mathbf{B}_P = \frac{2}{3} \mu_0 \sigma \omega R \mathbf{e}_z \]

and for \( z_p < -R \):

\[ \mathbf{B}_P = \frac{2 \mu_0 \sigma \omega R^4}{-3.z_p^3} \mathbf{e}_z \]

\( \mathbf{B} \) directed along \(+\mathbf{e}_z\) for all points everywhere on Z-axis !!

inside sphere: constant field !!

B-field of a rotating charged conductor
Plot of $B$ for:

- $Q = 1$
- $\mu_0 = 1$
- $\omega = 1$
- $R = 1$

(in SI-units)

$Q = \sigma.4\pi R^2$

**Conclusion:** inside conducting sphere: **on-axis**: field = constant.

**Question:** what about the field inside the sphere, but **off-axis**?

To be investigated in part II   === >
Conclusions for on-axis (1)

Conducting sphere

\[ |z_p| > R \]

\[ B = \frac{\mu_0 Q \omega R^2}{6\pi |z_p|^3} \mathbf{e}_z \]

\[ |z_p| < R \]

\[ B = \frac{\mu_0 Q \omega}{6\pi R} \mathbf{e}_z \]

Homogeneously charged sphere

(see other presentation)

\[ |z_p| > R \]

\[ B = \frac{\mu_0 Q \omega R^2}{10\pi |z_p|^3} \mathbf{e}_z \]

\[ |z_p| < R \]

\[ B = \frac{\mu_0 Q \omega}{20\pi R^3} \left(5R^2 - 3|z_p|^2\right) \mathbf{e}_z \]

\[ Q = \sigma \cdot 4\pi R^2 \]
Conclusions for on-axis (2)

Plot of $B$ for:

- $Q = 1$
- $\mu_0 = 1$
- $\omega = 1$
- $R = 1$

(in SI-units)

Homogeneously charged sphere

Conducting sphere

B-field of a rotating charged conducting sphere
Off-axis: Analysis and Symmetry (1)

Part II. Calculate $B$-field in point $P$ off the axis of rotation (Z-axis) inside or outside the sphere.

Rotation axis (Z-axis) =

= symmetry axis.

Assume $P (0, y_P, z_P)$ in YZ-plane.

Coordinate systems:
- $X, Y, Z$
- $r, \theta, \varphi$
Conducting sphere: off-axis (1)

Ring on surface of the sphere.

\[ dB = \frac{\mu_0 j' \times e_r}{4\pi r_p^2} dA \]

with \( j' \) in [A/m]

Surface charge \( \sigma.dA \) on \( dA \) will rotate with \( \omega \)

\[ db = R.d\theta \]

\[ dl = R \cdot \sin \theta \cdot d\varphi \]

\( dA = \text{width } db \cdot \text{length } dl \)

Needed:
- \( j' \), \( e_r \), \( r_p \)
Conducting sphere: off-axis (2)

\[ dB = \frac{\mu_0 j' \times e_r}{4\pi r^2} \]
\[ dA = \frac{\mu_0 j' \times r}{4\pi r^3} \]

\[ dA = R.d\theta. R.\sin\theta. d\phi \]

\[ j' = \sigma \omega R \sin \theta \]

Off-axis: \( j' \) not \( \perp e_r \) !!

\[ j' = j'(-\sin\phi.e_x + \cos\phi.e_y + 0.e_z) \]

\[ r = r_P - r_A \; ; \; A = \text{at } dA\text{-element} \]

\[ r_P = (0.e_x + y_P.e_y + z_P.e_z) \]

\[ r_A = R(\sin\theta.\sin\phi.e_x + \sin\theta.\cos\phi.e_y + \cos\theta.e_z) \]

B-field of a rotating charged conducting sphere.
Conducting sphere: off-axis (2)

\[ dB = \frac{\mu_0 j' \times e_r}{4\pi r^2} \, dA \]
\[ dA = \frac{\mu_0 j' \times r}{4\pi r^3} \, dA. \]

Analytical approach: not feasible (due to \( j \times r \) and \( r^3 \))

Numerical approach necessary.

\[ r_A = R (\sin \theta \sin \phi \, e_x + \sin \theta \cos \phi \, e_y + \cos \theta \, e_z) \]
Conducting sphere: off-axis (4)

Available for download on [www.demul.net/frits](http://www.demul.net/frits):
offline program: **EM_solenoids**
in file: **EM_programs.zzz**
on subpage **Electromagnetism**

This program can calculate:

* $B$- and $A$-fields for:
  * Single solenoids
  * Pairs of solenoids
  * Dipole fields
  * Field of a rotating charged conducting sphere
  * and sphere segments
Examples

1. Rotating charged conducting sphere

Properties:

- Charge = 1 C
- Radius = 5 cm
- Velocity = 1 rad/s = 0.1592 rev./s

NB. Rotation axis = symmetry axis = X-axis;
Fields shown in XY-plane at Z=0.
## Examples

1. Rotating charged conducting sphere: settings:

### Options
- Single solenoid
- Pair of solenoids
- Dipolar far-field approximation
- Conducting sphere or sphere segment

### Field pattern
- \( B : \) Field line vectors
- \( B : \) Modulus, values
- \( B : \) Modulus, squares
- \( B : \) X-components, values
- \( B : \) X-components, squares
- \( B : \) Y-components, values
- \( B : \) Y-components, squares
- \( A : \) Z-components, values
- \( A : \) Z-components, squares

### X-axis = rotation (symmetry) axis
- Sphere radius [cm] (center in 0): 5.00
- Charge: 1.000E+00
- Charge density on sphere/segment: 3.183E+01 C/m\(^2\)
- Rotation velocity [revolutions/sec]: 1.592E-01
- Idem: 1.000E+00 rad/s
- Current density at YZ-equator plane (X=0): 1.592E+00 A/m

### Polar angle [deg]:
- min: 0.00, max: 180.00
- Angular interval on sphere [deg]: 2.00

### Plot dimensions [cm]
- X left: -10.0, X right: 10.0
- Y low: -10.0, Y high: 10.0
- Interval: 1.0, Font size (px): 12
**B-field:**
Sphere rotating around X-axis

Inside the sphere:
homogeneous field

Field strength inside = 1.3337 \( \mu \)T
B-field: Sphere rotating around X-axis

Inside the sphere: homogeneous field

Field strength inside = 1.3337 μT
Conducting sphere rotating around X-axis: $B$ and $A$-fields

**B-field:** Cross section of sphere: XY-plane at $Z=0$:

Inside the sphere:
Homogeneous $B$-field $\Rightarrow$ $A$-field varies linearly with $y$-coordinate (due to derivatives in $\text{rot} (\text{curl})$)

Expression for a surface current:

$$dB = \frac{\mu_0 j' \times e_r}{4\pi} \frac{1}{r^2} dA$$

**A-field:** Vector potential:

$$B = \text{rot} A (= \text{curl} A)$$

$$dA = \frac{\mu_0 j'}{4\pi} \frac{1}{r} dA$$

$B$ and $A$: perpendicular fields.

For points in XY-plane:
$B$ in XY-plane, no $Z$-component
$A \bot$ XY-plane, $Z$-component only.

For points outside XY-plane:
Cylindrical symmetry around X-axis.
A-field: Sphere rotating around X-axis

A = 0 at rotation symmetry axis
Examples

2. Rotating charged conducting sphere segment between $45^0$ and $135^0$ (ring shape)

Properties:

- Charge = 1 C
- Radius = 5 cm
- Velocity = 1 rad/s = 0.1592 rev./s
**B-field:**

Sphere segment (ring shape) rotating around X-axis

Field already looks like a solenoid field
B-field: Sphere segment (ring shape) rotating around X-axis

Field already looks like a solenoid field
3. Rotating charged conducting sphere segment between $120^0$ and $180^0$ (bowl shape)

Properties:

- Charge = $1 \text{ C}$
- Radius = $5 \text{ cm}$
- Velocity = $1 \text{ rad/s} = 0.1592 \text{ rev./s}$
B-field: Sphere segment (bowl shape) rotating around X-axis

Field already looks like a dipolar field
B-field: Sphere segment (bowl shape) rotating around X-axis

Field already looks like a dipolar field
the end