University of Twente
Department Applied Physics

First-year course on

Electromagnetism

Magnetism: topics

Version 2: Sept. 2017

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Electromagnetism

Magnetism: topics

Contents:

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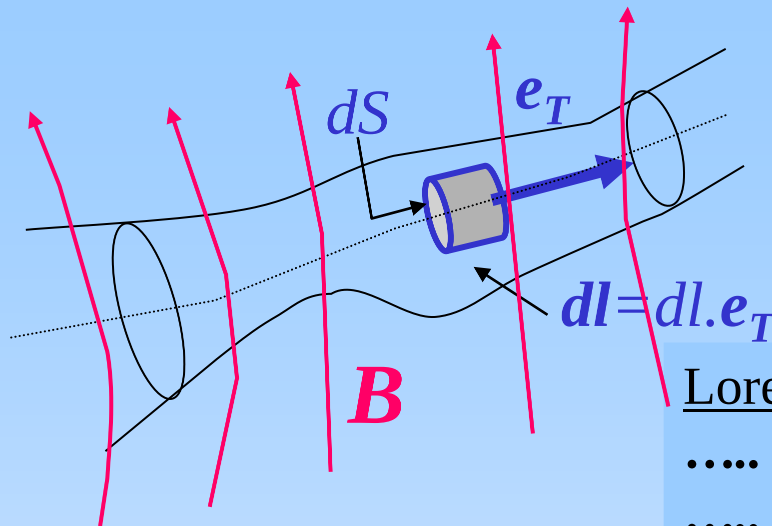
1. Magnetic Force on a Current Wire



Suppose: total current = I ;
cross section S variable

$|\boldsymbol{j}| = dI/dS$
$\boldsymbol{j} = n\, q\, \boldsymbol{v}$ (n = # particles/m^3)

Lorentz force on one charge: $\boldsymbol{F} = q\, \boldsymbol{v} \times \boldsymbol{B}$

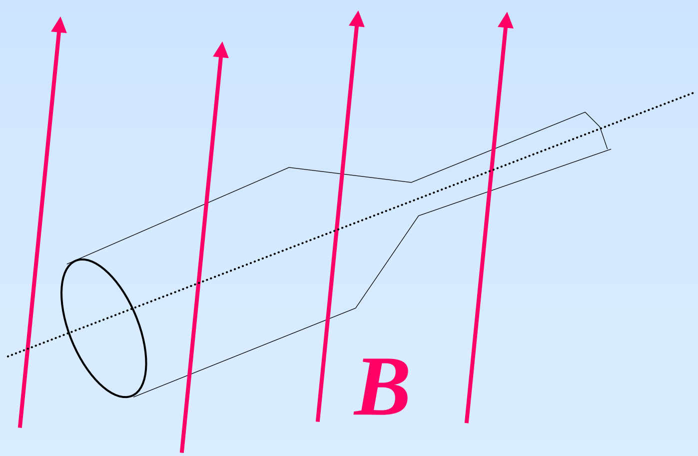
….. per unit of volume : $\boldsymbol{f} = nq\, \boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{j} \times \boldsymbol{B}$

….. on volume dV : $\boldsymbol{dF} = \boldsymbol{j} \times \boldsymbol{B}.\, dV$

$\boldsymbol{dF} = \boldsymbol{j} \times \boldsymbol{B}.\, dS.dl$

$\boldsymbol{j} = j.\boldsymbol{e_T}$ $\qquad \boldsymbol{dF} = j.dS.\, \boldsymbol{e_T} \times \boldsymbol{B}\, .\, dl$

$$\boldsymbol{dF} = I\, .\, \boldsymbol{dl} \times \boldsymbol{B}$$



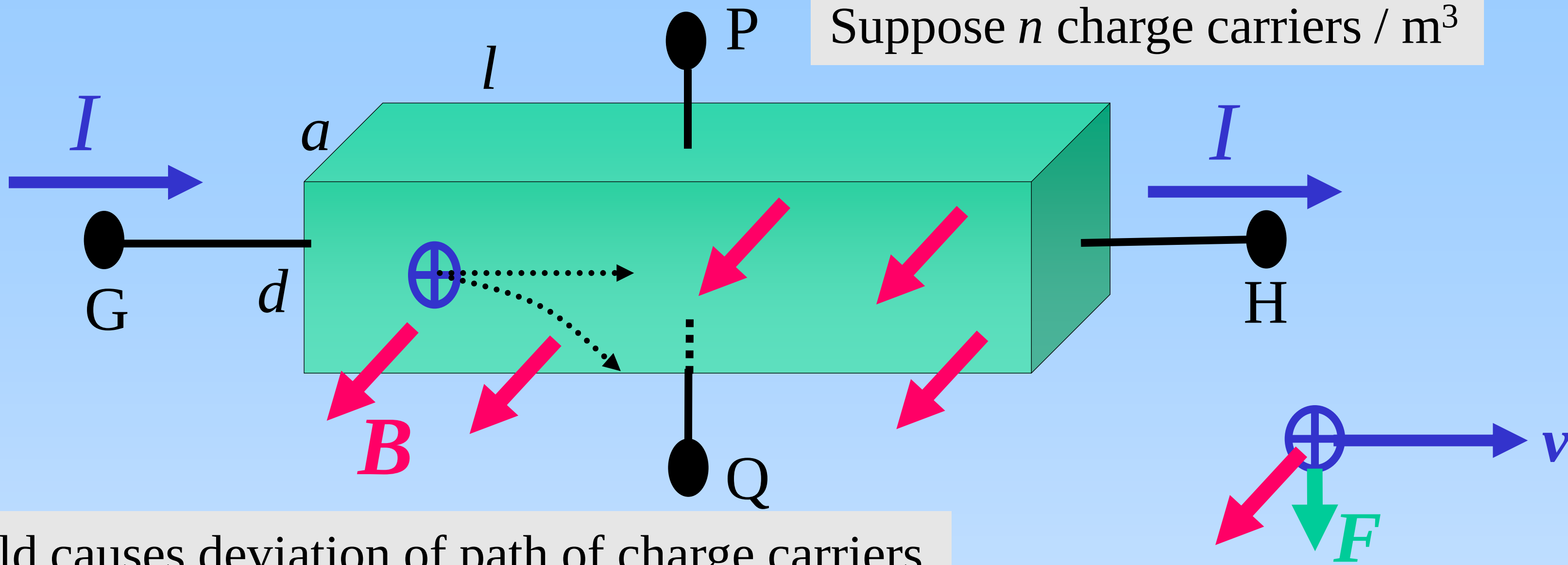
Straight conductor in homogeneous field:

$\boldsymbol{F} = I.\boldsymbol{e_T} \times \boldsymbol{B} \int dl =$ $\quad \boldsymbol{F} = I.\, L.\, \boldsymbol{e_T} \times \boldsymbol{B}$

$\boldsymbol{F}$ pointing ⊥ plane of drawing

2. Hall effect

Suppose n charge carriers / m^3



B-field causes deviation of path of charge carriers

Build up of electric field $\boldsymbol{E}_{Hall}$ between Q and P: Q+ ; P-

Stationary case: $\boldsymbol{F}_{magn} = \boldsymbol{F}_{elec} \Rightarrow q\,v\,B = q\,E_{Hall}$

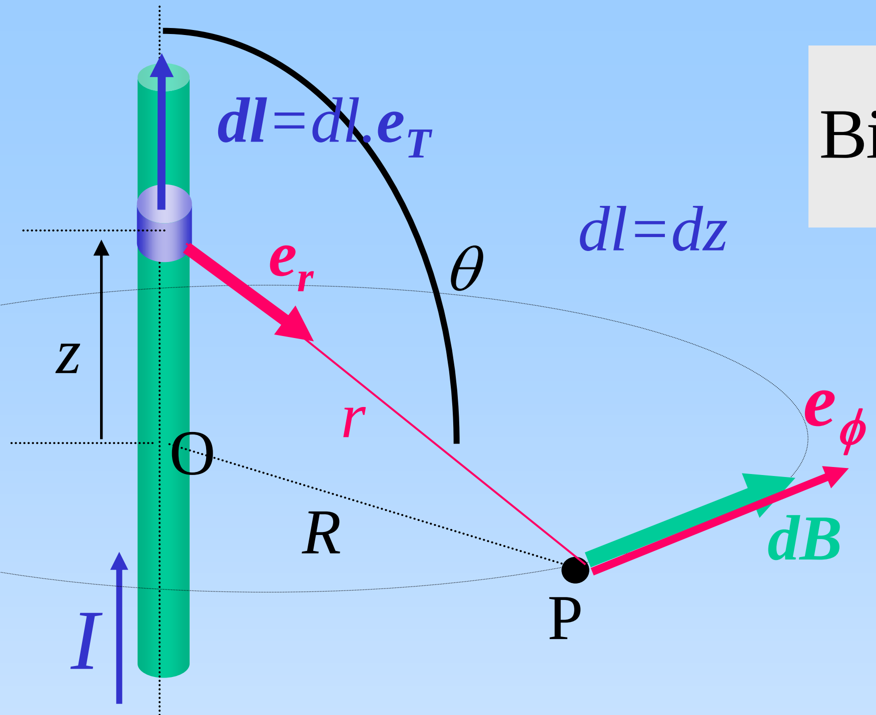
With $j = nqv = I/(ad) \Rightarrow$

$$E_{Hall} = vB = \frac{IB}{nqad}$$

$$\text{Hall potential}: V_{Hall} = V_Q - V_P = E_{Hall}.\,d = \frac{IB}{nqa}$$

$V_{Hall} \sim B$
magnetometer

3. Magnetic field of a line current



Biot & Savart : $\boldsymbol{dB} = \dfrac{\mu_0 I}{4\pi} \dfrac{\boldsymbol{e_T} \times \boldsymbol{e_r}}{r^2} dl$

<u>Question</u>: Determine ***B*** in P

<u>Approach</u>: Current line elements *dl*

<u>Calculation:</u> $\boldsymbol{e_T}\ x\ \boldsymbol{e_r} = \boldsymbol{e_\phi}$;
tangential component only: ***dB***

$$dB = \frac{\mu_0 I}{4\pi} \frac{\sin\theta}{r^2} dz$$

$$(1)\ \text{var} = \text{z}: \quad dB = \frac{\mu_0 I}{4\pi} \frac{1}{R^2 + z^2} \frac{R}{\sqrt{R^2 + z^2}} dz$$

Integration over *z* from -∞ to +∞

$$(2)\ \text{var} = \theta: \quad dB = \frac{\mu_0 I}{4\pi} \frac{1}{R^2 \sin^{-2}\theta} \sin\theta \frac{R d\theta}{\sin^2\theta}$$

Integration over θ from 0 to π, with z/R =-tan θ

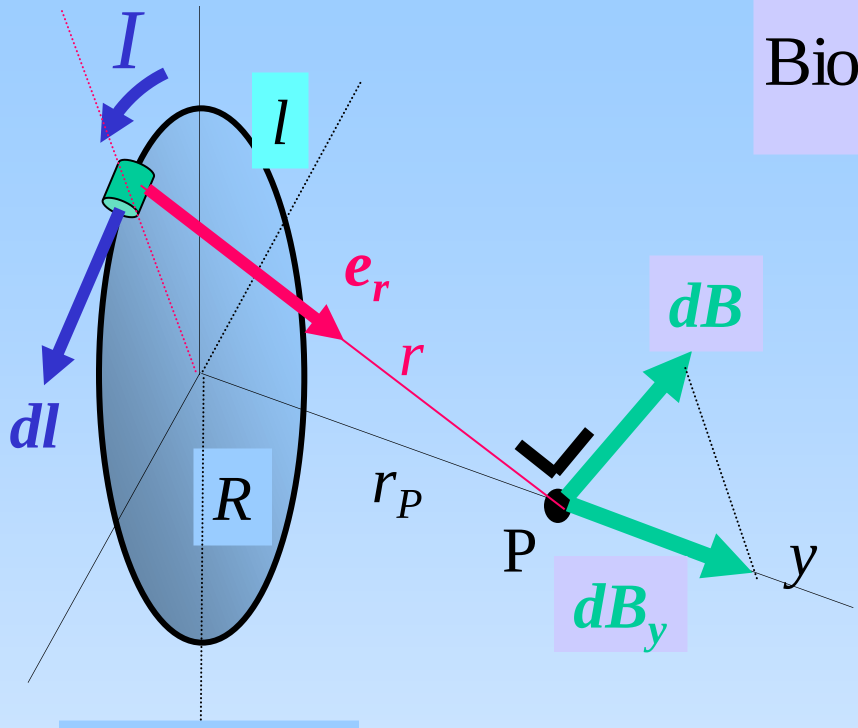
Result :

$$\boldsymbol{B}_P = \frac{\mu_0 I}{2\pi R} \boldsymbol{e}_\varphi$$

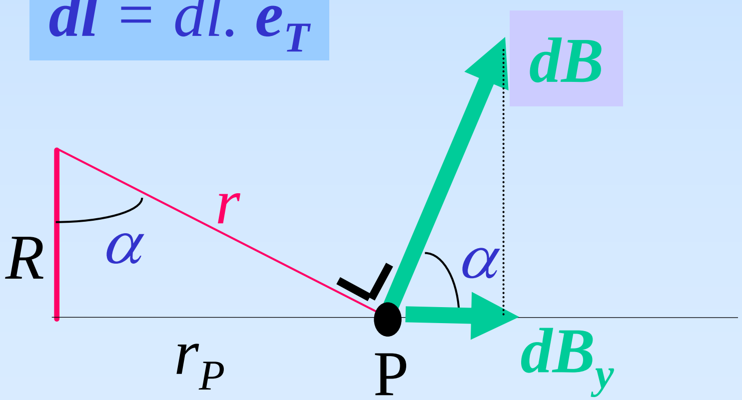
B ~ 1/*R*: cylinder symmetry

FdM

4. Magnetic field of a circular circuit



$\boldsymbol{dl} = dl.\,\boldsymbol{e_T}$



Biot & Savart: $\boldsymbol{dB} = \dfrac{\mu_0 I}{4\pi} \dfrac{\boldsymbol{e_T} \times \boldsymbol{e_r}}{r^2} dl$

Question: Determine $\boldsymbol{B}$ in P

Approach: Current line elements dl

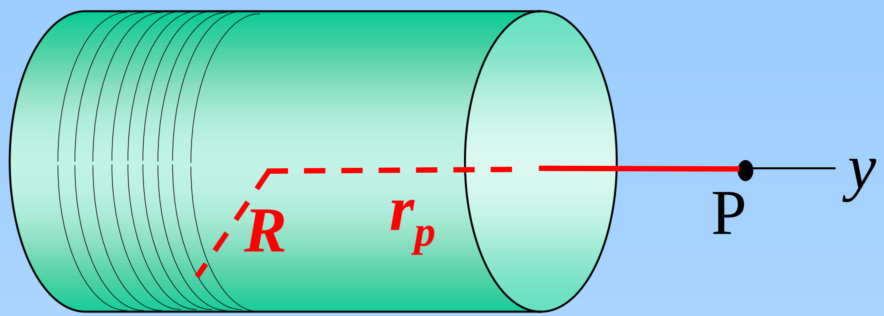
Calculation: $\boldsymbol{e_T} \times \boldsymbol{e_r} = 1$;
symmetry: $\boldsymbol{y}$- component only:

$$dB_y = \frac{\mu_0 I}{4\pi} \oint_l \frac{1}{r^2} dl \cos\alpha$$

$$dB_y = \frac{\mu_0 I}{4\pi} \frac{1}{r_p{}^2 + R^2} 2\pi R \frac{R}{\sqrt{r_p{}^2 + R^2}}$$

$$\boldsymbol{B}_P = \frac{\mu_0 I . R^2}{2\left(r_p{}^2 + R^2\right)^{3/2}} \boldsymbol{e_y}$$

5. Magnetic field of a circular solenoid



Radius: R ; Current I
Length: L
Coils: N , or per meter: n

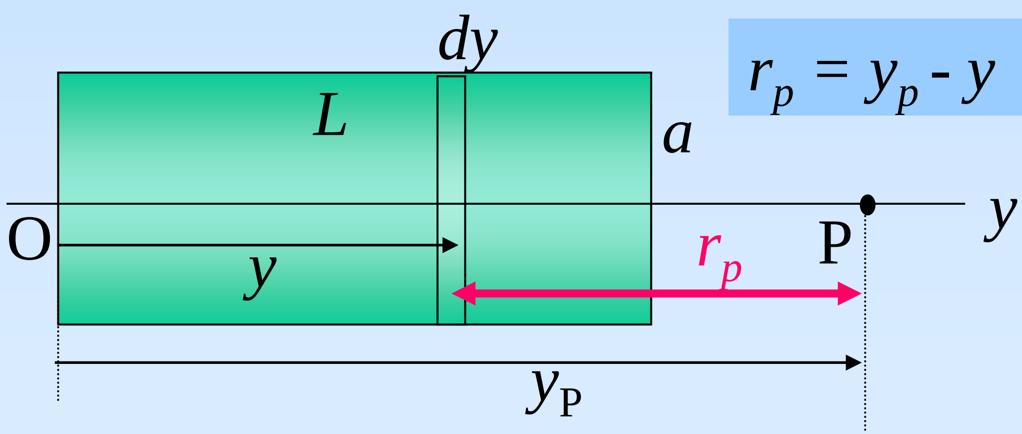
Question: Determine $\boldsymbol{B}$ in P

Approach: Solenoid = set of circular circuits ; and for each circuit:

$$\boldsymbol{B}_P = \frac{\mu_0 I . R^2}{2\left(r_p{}^2 + R^2\right)^{3/2}} \boldsymbol{e}_y$$

r_p is distance from circuit to P

Each circuit: strip dy; current $dI = n.dy.I$



$r_p = y_p - y$

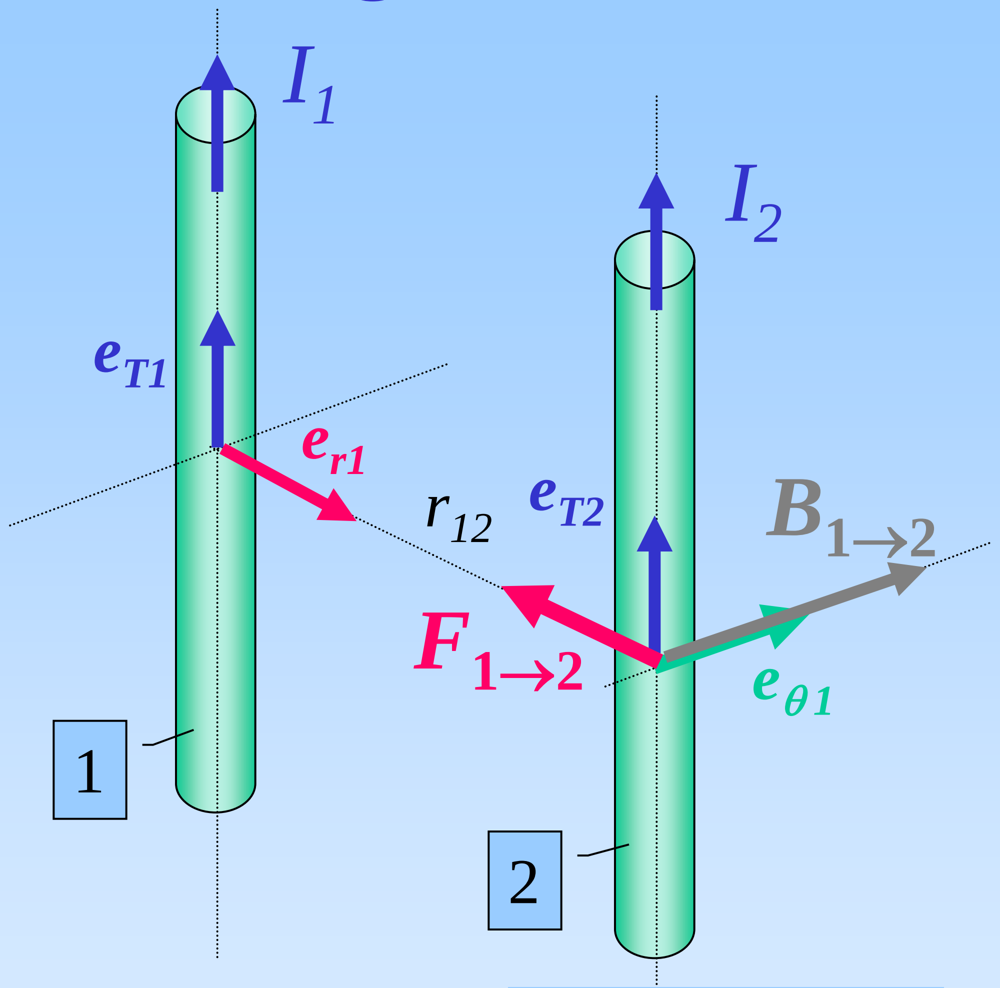
$$\boldsymbol{B} = \int_0^L \frac{\mu_0.(nI.dy).R^2}{2\left(r_p{}^2 + R^2\right)^{3/2}} \boldsymbol{e}_y$$

Result for $L \to \infty$:

$$\boldsymbol{B} = \mu_0\, n\, I\, \boldsymbol{e}_y$$

Result independent of R, L

6. Magnetic forces between currents



<u>Question:</u> determine $\boldsymbol{F}_{1\to 2}$ (force exerted by 1 on 2) :

Relations:

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi R}\boldsymbol{e}_\theta \quad ; \quad \boldsymbol{e}_\theta = \boldsymbol{e}_T \times \boldsymbol{e}_r$$

$$\boldsymbol{dF}_L = I\,\boldsymbol{dl} \times \boldsymbol{B} \quad ; \quad \boldsymbol{dl} = dl.\boldsymbol{e}_T$$

<u>Calculation</u>

$$\boldsymbol{B}_{1\to 2} = \frac{\mu_0 I_1}{2\pi r_{12}}\boldsymbol{e}_{\theta 1}$$

$$\boldsymbol{F}_{1\to 2} = \int I_2.\boldsymbol{e}_{T2} \times \boldsymbol{B}_{1\to 2}.dl_2$$

$$\boldsymbol{F}_{1\to 2} = \frac{\mu_0 I_1 I_2}{2\pi\, r_{12}} L_2(-\boldsymbol{e}_{r1})$$

$$\boldsymbol{e}_{T2} \times \boldsymbol{e}_{\theta 1} = -\boldsymbol{e}_{r1}$$

$$\text{If } L_1 = L_2 \quad : \quad \boldsymbol{F}_{1\to 2} = -\boldsymbol{F}_{2\to 1}$$

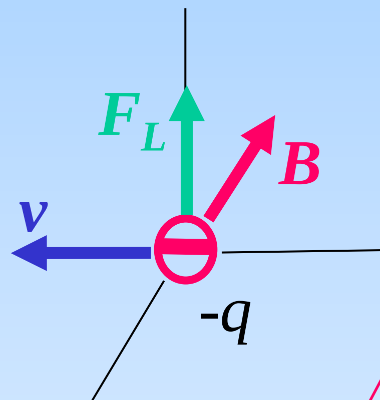
If currents have same direction: force **attractive**

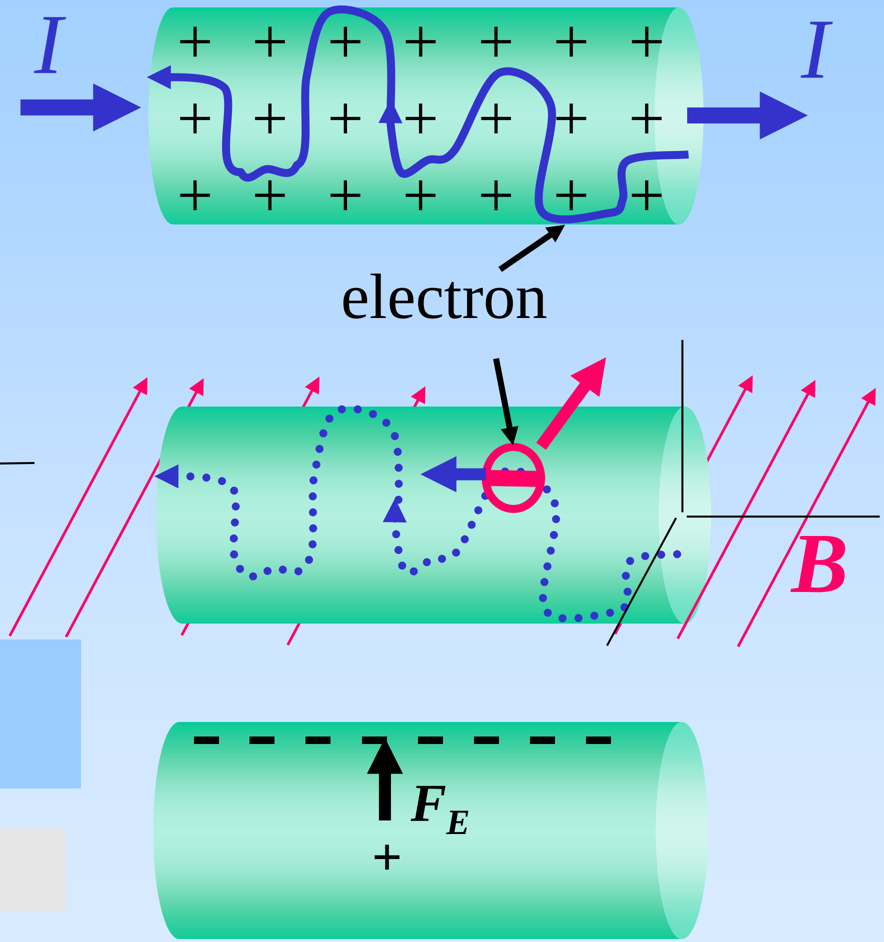
7. Why is the **wire** moved by Lorentz force ?

(since inside the wire only the conduction electrons move, and not the metal ions)

Conductor:
- fixed ion lattice,
- conduction electrons

Magnetic field ***B***
⊥ plane of drawing



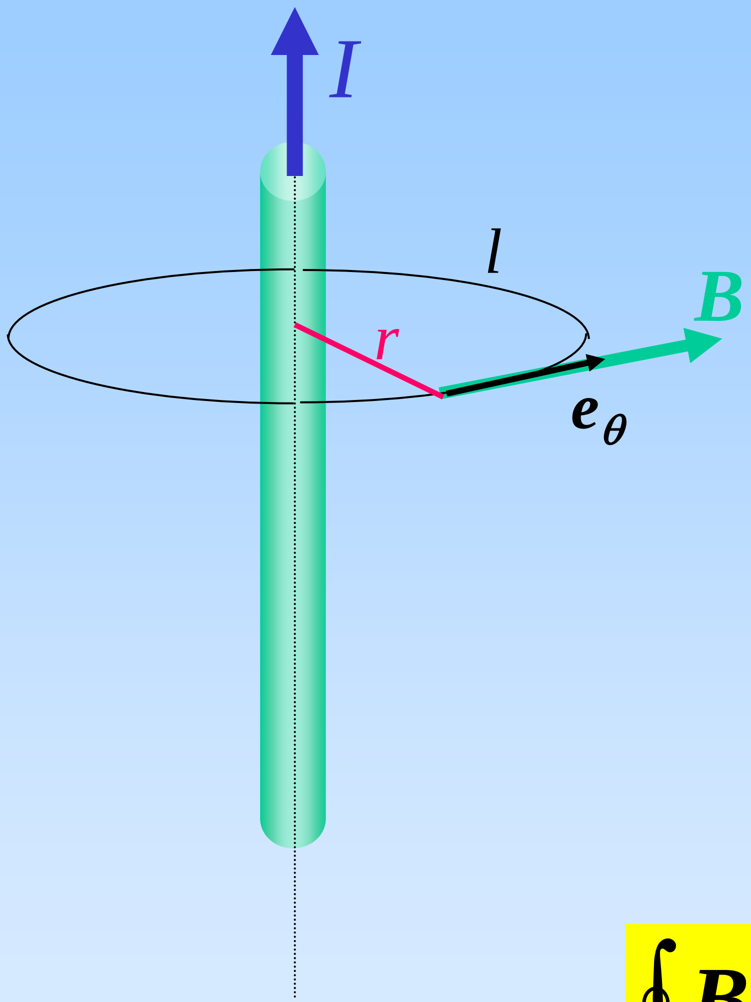


Hall effect: concentration of electrons (-charge) at one side of conductor

Lattice ions feel a force $\boldsymbol{F_E}$ upwards

This force (= $\boldsymbol{F_L}$) is electric !!

8. Ampère's Law (1)



Long thin straight wire; current I

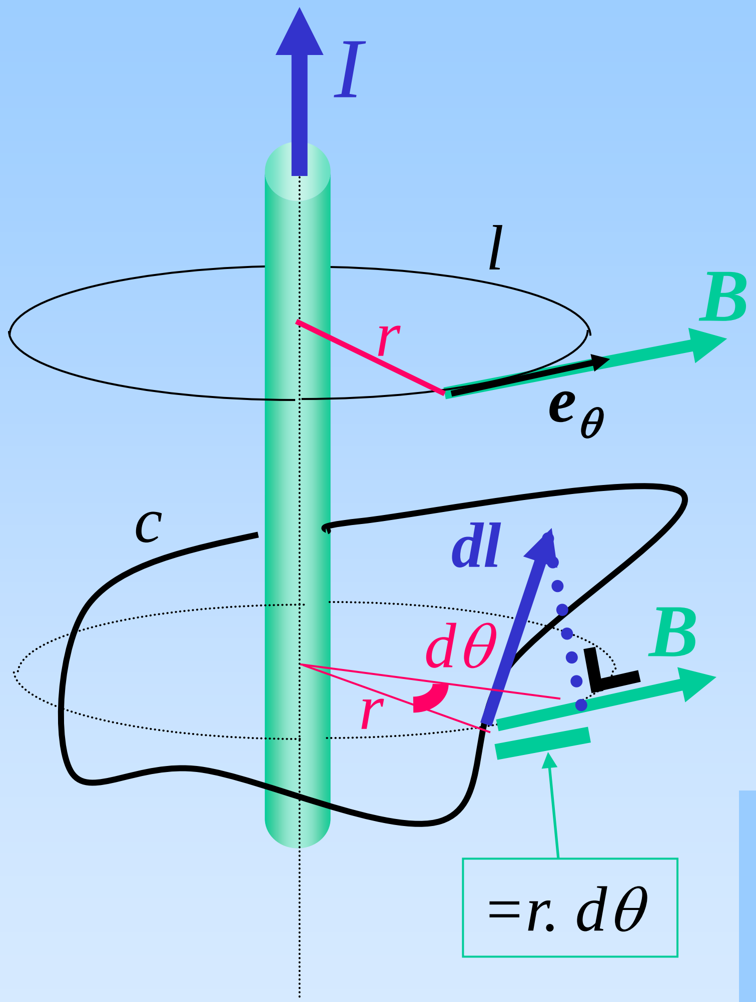
Question: Determine the "Circulation of $\boldsymbol{B}$-field" along circle l $\oint \boldsymbol{B} \bullet \boldsymbol{dl}$

With Biot & Savart: $\boldsymbol{B} = \dfrac{\mu_0 I}{2\pi r}\boldsymbol{e}_\theta$

$$\oint_l \boldsymbol{B} \bullet \boldsymbol{dl} = \oint_l \frac{\mu_0 I}{2\pi r}\boldsymbol{e}_\theta \bullet \boldsymbol{e}_\theta \, dl = \frac{\mu_0 I}{2\pi r} 2\pi r$$

$$\oint_l \boldsymbol{B} \bullet \boldsymbol{dl} = \mu_0 I \quad \text{: Ampere's Law}$$

8. Ampère's Law (2)



$\oint_l \boldsymbol{B}\bullet \boldsymbol{dl} = \mu_0 I$: Ampere

Question: Determine the "Circulation of ***B***-field" $\oint \boldsymbol{B}\bullet \boldsymbol{dl}$ along circuit *c*

$$\oint_c \boldsymbol{B}\bullet \boldsymbol{dl} = \oint_c B.r.d\theta = \frac{\mu_0 I}{2\pi r}\, r.2\pi$$

and again : $\oint_c \boldsymbol{B}\bullet \boldsymbol{dl} = \mu_0 I$

Consequences:

1. More currents through *c* add up ;
2. Currents outside *c* do not contribute ;
3. Position of current inside *c* is not important.

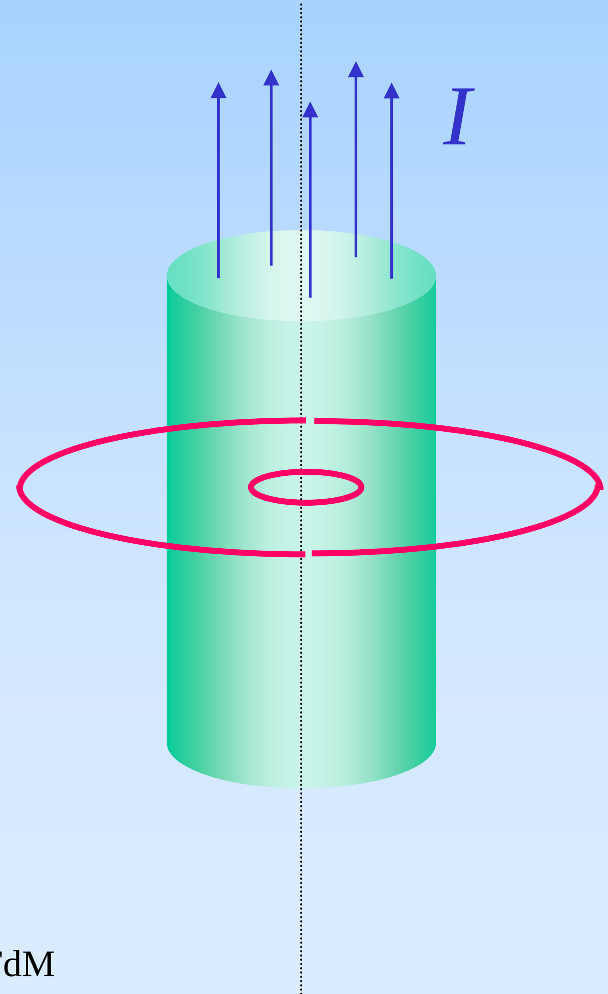
9. ***B***-field from a thick wire

<u>Cylinder:</u> radius R

current : $I = \iint_S \boldsymbol{j} \bullet d\boldsymbol{S}$

$$\oint_l \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_0 I = \mu_0 \iint_S \boldsymbol{j} \bullet d\boldsymbol{S}$$

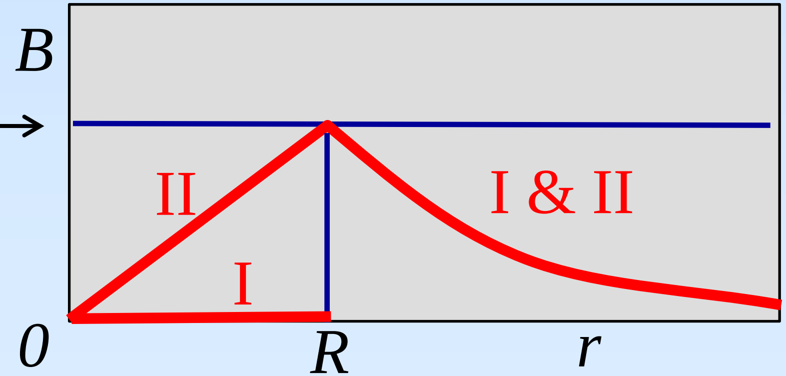
<u>Options for current</u>:
I: at surface II: in volume
(suppose: homogeneous)



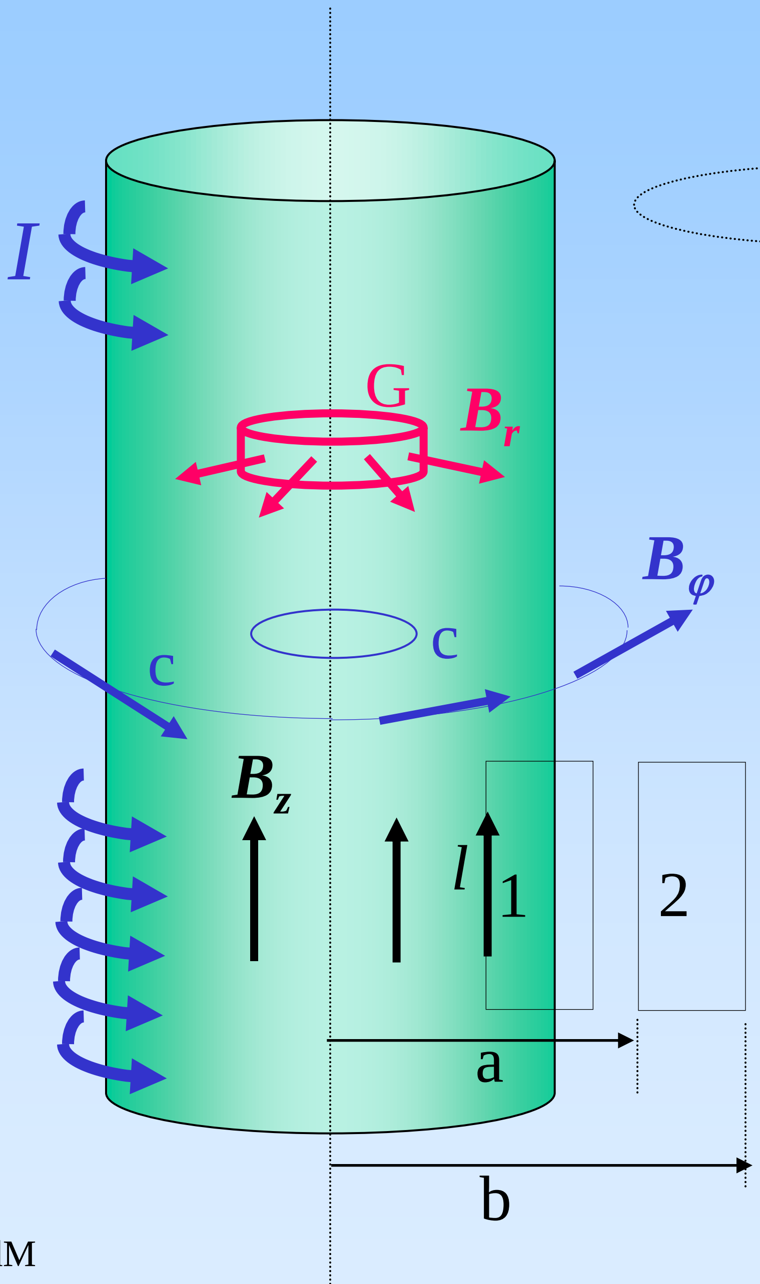
Use Ampere-circuits
(radius r):

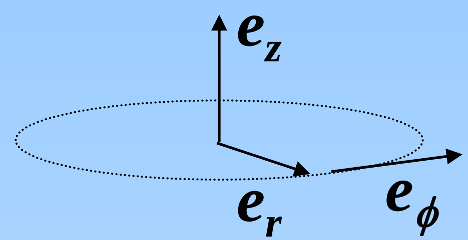
$r \geq R$	$B(r).2\pi r = \mu_0 I$	$B(r) = \dfrac{\mu_0 I}{2\pi r}$
$r \leq R$	$(I): B(r).2\pi r = 0$	$B(r) = 0$
	$(II): B(r).2\pi r = \mu_0 I \dfrac{\pi r^2}{\pi R^2}$	$B(r) = \dfrac{\mu_0 I.r}{2\pi R^2}$

$\dfrac{\mu_0 I}{2\pi R} \longrightarrow$



10. Magnetic Induction of a Solenoid





Radius: R ; Current: I
Length: $L \quad >> R$
Coils: n per meter

Components: $\boldsymbol{B_z}\ \boldsymbol{B_r}\ \boldsymbol{B_\phi}$

B_r: Gauss-box G: $\quad \Phi_{total}=0$
$\Phi_{top} = \Phi_{bottom} \Rightarrow \Phi_{wall}=0 \Rightarrow B_r=0$

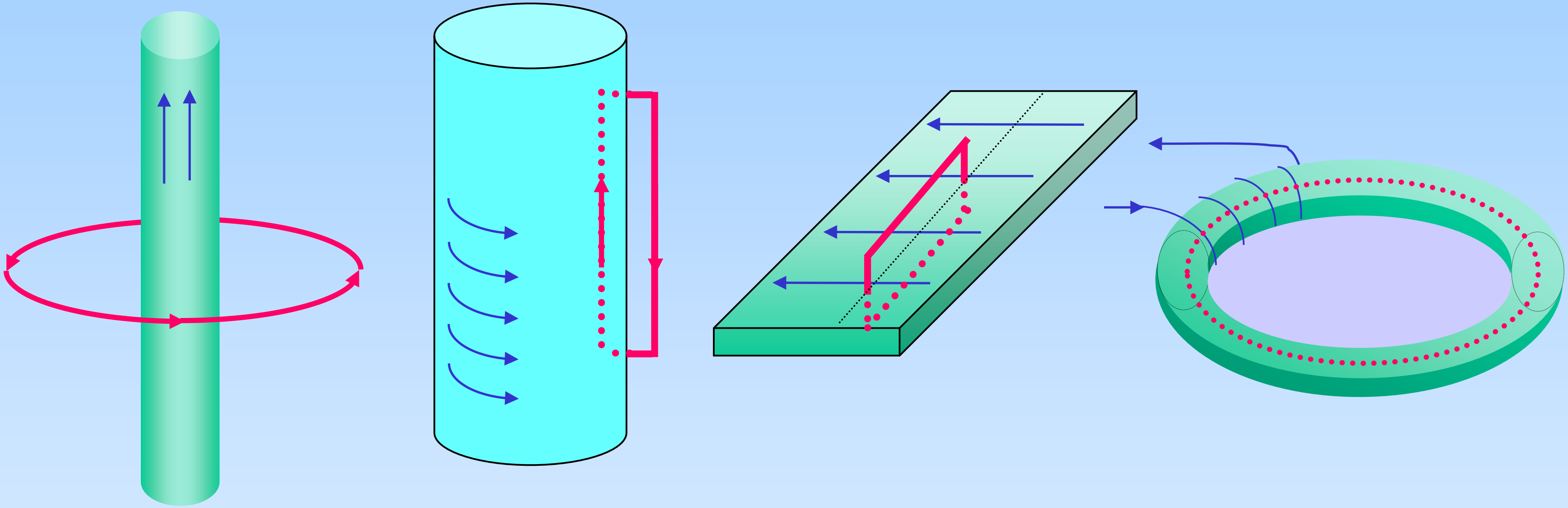
B_ϕ: Circuit c (radius r):
Ampere: $B_\phi \,.2\pi r=0 \Rightarrow B_\phi=0$

B_z: Circuit 2: $B(a)=B(b)=0$
Circuit 1: Ampere: $B_z\, l=\mu_0\, n\, I\, l$

Result: inside: $\boldsymbol{B} = \mu_0\, n\, I\, \boldsymbol{e_z}$
outside: $\boldsymbol{B} = 0$

11. Symmetries for Ampere's Law

$$\oint_l \mathbf{B} \bullet \mathbf{dl} = \mu_0 I = \mu_0 \iint_S \mathbf{j} \bullet \mathbf{dS}$$



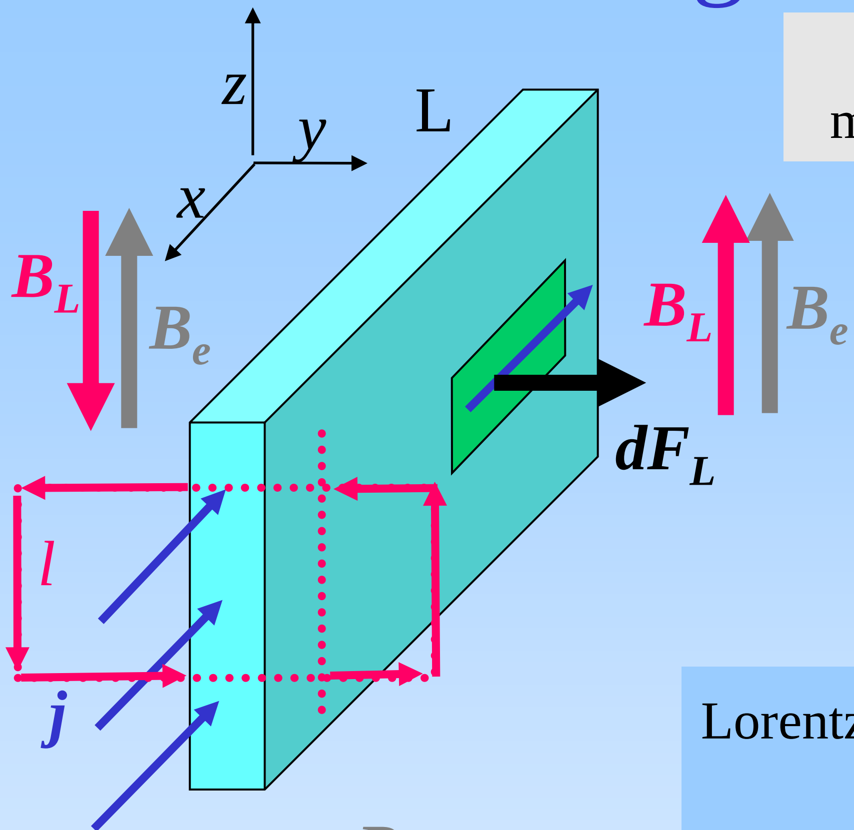
Wire,
∞ long

Solenoid,
∞ long

Plane,
∞ extending

Toroid,
along core line

12.Magnetic Pressure



<u>Question:</u> why does a <u>solenoid</u> try to maximize its cross section ("fitting flux") ?

Plane layer L with current density $\boldsymbol{j}$

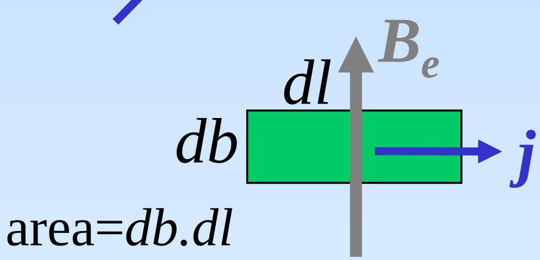
$\boldsymbol{B}$-field of the layer (circuit l):
$2\,B_L\; l = \mu_0\, j\; l \;\Rightarrow\; \boldsymbol{B_L} = \pm\tfrac{1}{2}\mu_0\, j\; \boldsymbol{e_z}$

Suppose we add an external field $\boldsymbol{B_e}$, with $B_e = B_L$, so that the total field <u>behind</u> the layer = 0

Lorentz force on : ($\boldsymbol{F_L}$=$I.L.\boldsymbol{e_T} \times \boldsymbol{B_e}$):

$\boldsymbol{dF_L}$= ($\boldsymbol{j}.db).dl \times \tfrac{1}{2}\mu_0\, j\; \boldsymbol{e_z}$

$\boldsymbol{dF_L}$= $\tfrac{1}{2}\mu_0\, j^2.db.dl\; \boldsymbol{e_y}$



Magnetic pressure: $P = \tfrac{1}{2}\mu_0\, j^2 = \tfrac{1}{2}B_e^{\,2}/\mu_0$.

<u>Example</u>: this situation is met at the wall of a (long) solenoid. Then the pressure is outward, thus maximizing the cross section area.

13. Vector potential *A*

Electric (scalar) potential : $\boldsymbol{E} = -\nabla V$

Magnetic (vector) potential : $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

$$\boldsymbol{E} = -\left(\frac{\partial V}{\partial x} \boldsymbol{e}_x + \frac{\partial V}{\partial y} \boldsymbol{e}_y + \frac{\partial V}{\partial z} \boldsymbol{e}_z \right)$$

$$\boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

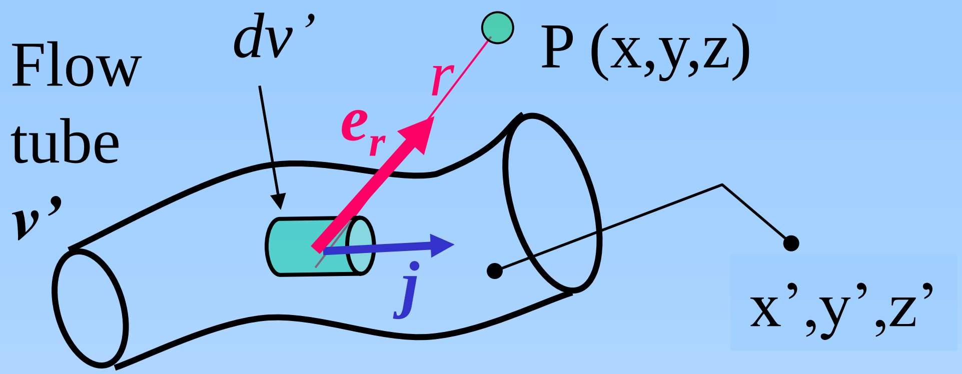
Only the **spatial <u>derivatives</u>** of *V* and ***A*** are defined !

The **absolute values** of *V* and ***A*** can be determined by integration, but up to a constant (integration) term.

Therefore, only **<u>potential differences</u>** of *V* and ***A*** between two points in space have a physical meaning !

One of these points may act as the "**reference point**".

13. Vector potential *A*



Definition of ***A***: $\boldsymbol{B} = \nabla \times \boldsymbol{A}$

Question: determine ***A*** in P from Biot-Savart for ***B***

$r = f(x,y,z,\ x',y',z')$!!!

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\iiint_{v'} \frac{\boldsymbol{j}\times\boldsymbol{e}_r}{r^2}\,dv' = \frac{\mu_0}{4\pi}\iiint_{v'} \boldsymbol{j}\times\left(-\nabla\frac{1}{r}\right)dv' = \frac{\mu_0}{4\pi}\iiint_{v'}\left(\nabla\frac{1}{r}\right)\times\boldsymbol{j}.dv'$$

$$u.dv = d(uv) - v.du$$

$$\Rightarrow \boldsymbol{B} = \frac{\mu_0}{4\pi}\iiint_{v'}\left[\nabla\times\left(\frac{\boldsymbol{j}}{r}\right) - \frac{\nabla\times\boldsymbol{j}}{r}\right]dv'$$

$$\nabla = f(x,y,z);\quad \boldsymbol{j} = f(x',y',z') \quad \Rightarrow \nabla\times\boldsymbol{j} = 0$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\iiint_{v'}\left[\nabla\times\left(\frac{\boldsymbol{j}}{r}\right)\right]dv'$$

Swap differentiation (=*f* (*xyz*)) and integration (=*f* (*x'y'z'*)) :

$$\Rightarrow \boldsymbol{B} = \nabla\times\left[\frac{\mu_0}{4\pi}\iiint_{v'}\frac{\boldsymbol{j}}{r}\,dv'\right]$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\iiint_{v'}\frac{\boldsymbol{j}}{r}\,dv'$$

13. Vector potential *A*

$$\text{general}: \quad \boldsymbol{B} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j} \times \boldsymbol{e_r}}{r^2} dv' \Rightarrow \boldsymbol{A} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j}}{r^2} dv'$$

$$\text{circuit} : \boldsymbol{B} = \frac{\mu_0 \boldsymbol{I}}{4\pi} \oint_{loop} \frac{\boldsymbol{e_r}}{r^2} dl \Rightarrow \boldsymbol{A} = \frac{\mu_0 \boldsymbol{I}}{4\pi} \oint_{loop} \frac{\boldsymbol{dl}}{r}$$

Helpful relations:

$$\oint_c \boldsymbol{A} \bullet \boldsymbol{dl} = \iint_S (\nabla \times \boldsymbol{A}) \bullet \boldsymbol{dS} = \iint_S \boldsymbol{B} \bullet \boldsymbol{dS} = \Phi$$

Circuit *c* encloses area *S* ; Stokes' relation

$$\text{with } \nabla \times \boldsymbol{B} = \nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \bullet \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

$$\text{and} \quad \nabla \bullet \boldsymbol{A} = 0 \Rightarrow \nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{j}$$

Poisson's equations

$$\text{NB}: \text{electric potential} : \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

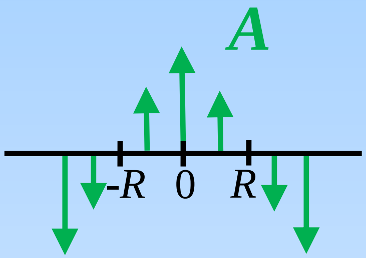
13. Vector potential $\boldsymbol{A}$: examples (1)

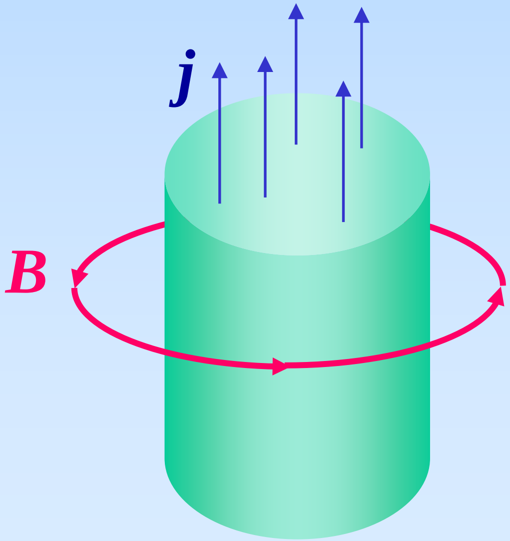
general : $\boldsymbol{B} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j} \times \boldsymbol{e}_r}{r^2} dv' \Rightarrow \boldsymbol{A} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j}}{r} dv'$

circuit : $\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_{loop} \frac{\boldsymbol{e}_r}{r^2} dl \Rightarrow \boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint_{loop} \frac{\boldsymbol{dl}}{r}$

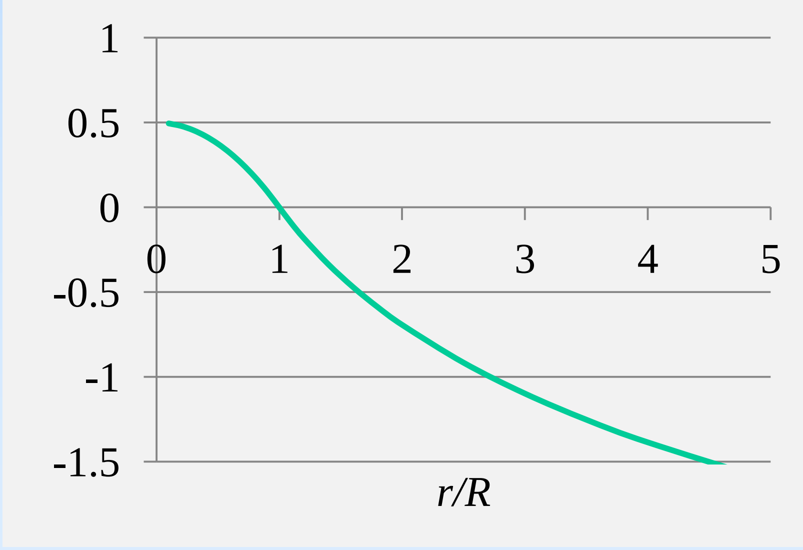
Wire,
radius R,
∞ long

Integration of $1/r$ to $\ln(r)$ leads to result $\rightarrow \infty$.
Therefore, point $r=R$ is used as the reference (value set to 0).





Plot: A(r) / [$\mu_0 I$ / (2π)]



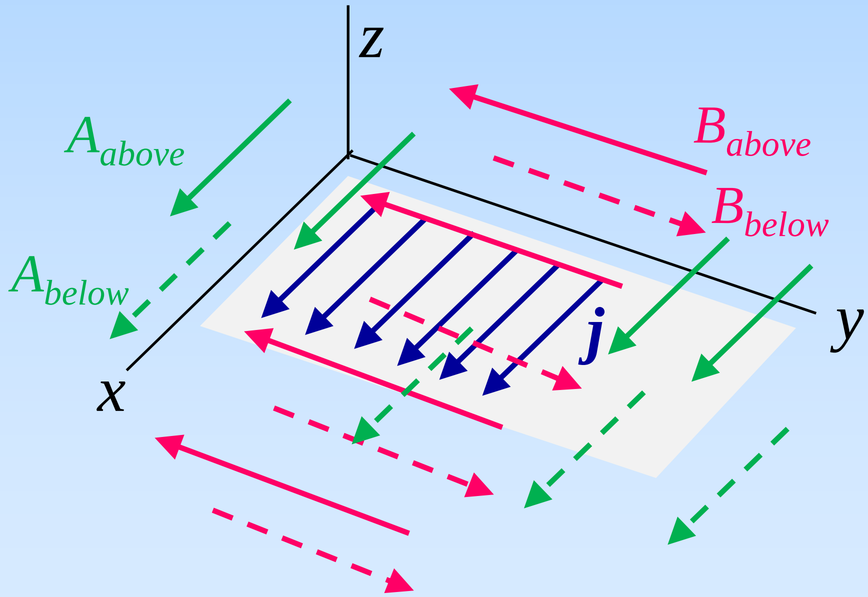
13. Vector potential *A* : examples (2)

general: $\boldsymbol{B} = \frac{\mu_0}{4\pi}\iiint_{v'} \frac{\boldsymbol{j}\times\boldsymbol{e}_r}{r^2}\,dv' \Rightarrow \boldsymbol{A} = \frac{\mu_0}{4\pi}\iiint_{v'} \frac{\boldsymbol{j}}{r}\,dv'$

circuit: $\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\oint_{loop} \frac{\boldsymbol{e}_r}{r^2}\,dl \Rightarrow \boldsymbol{A} = \frac{\mu_0 I}{4\pi}\oint_{loop} \frac{\boldsymbol{dl}}{r}$

Sheet

∞ long and wide



Result: (c = const. [A/m])

$\boldsymbol{j} = c\,\boldsymbol{e}_x$

$\boldsymbol{B} = \pm \tfrac{1}{2}\,\mu_0\,c\,\boldsymbol{e}_y$

$\boldsymbol{A} = \tfrac{1}{2}\,\mu_0\,c\,|z|\,\boldsymbol{e}_x$

A-values with respect to reference at z=0.

used: Stokes:

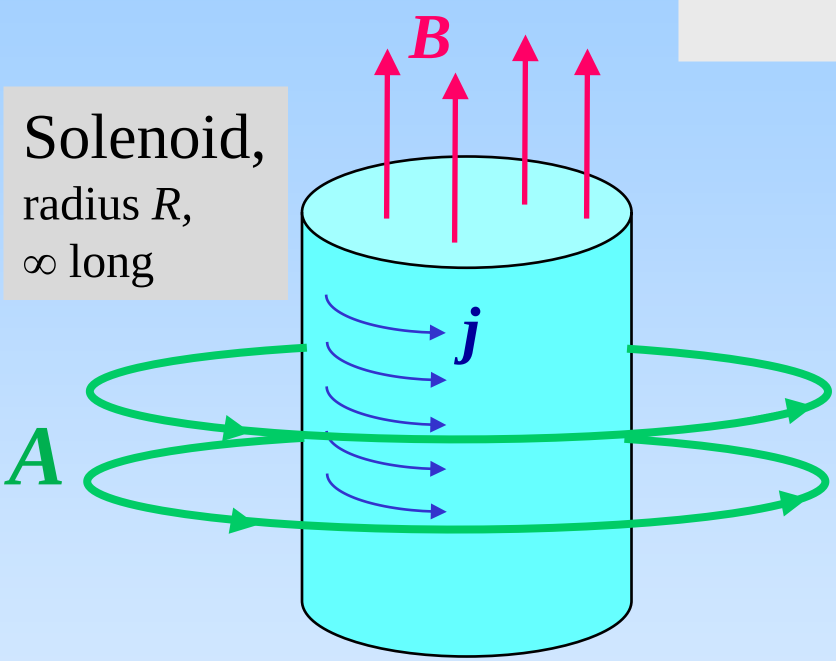
$\oint_c \boldsymbol{A}\bullet\boldsymbol{dl} = \iint_S (\nabla\times\boldsymbol{A})\bullet\boldsymbol{dS} = \iint_S \boldsymbol{B}\bullet\boldsymbol{dS} = \Phi$

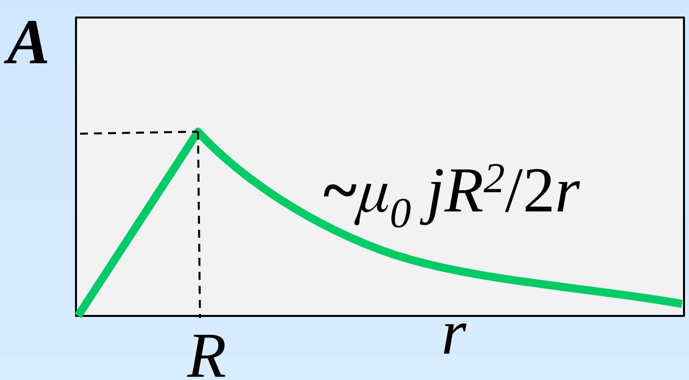
in a rectangle with sides // X- and Z-axes

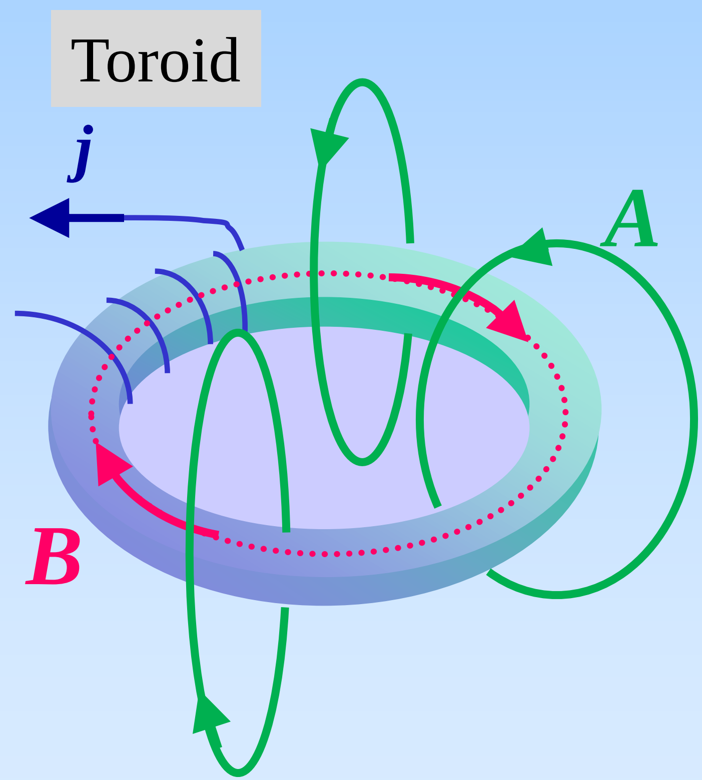
13. Vector potential *A* : examples (3)

general : $\boldsymbol{B} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j} \times \boldsymbol{e}_r}{r^2} dv' \quad \Rightarrow \quad \boldsymbol{A} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j}}{r} dv'$

circuit : $\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_{loop} \frac{\boldsymbol{e}_r}{r^2} dl \quad \Rightarrow \quad \boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint_{loop} \frac{\boldsymbol{dl}}{r}$





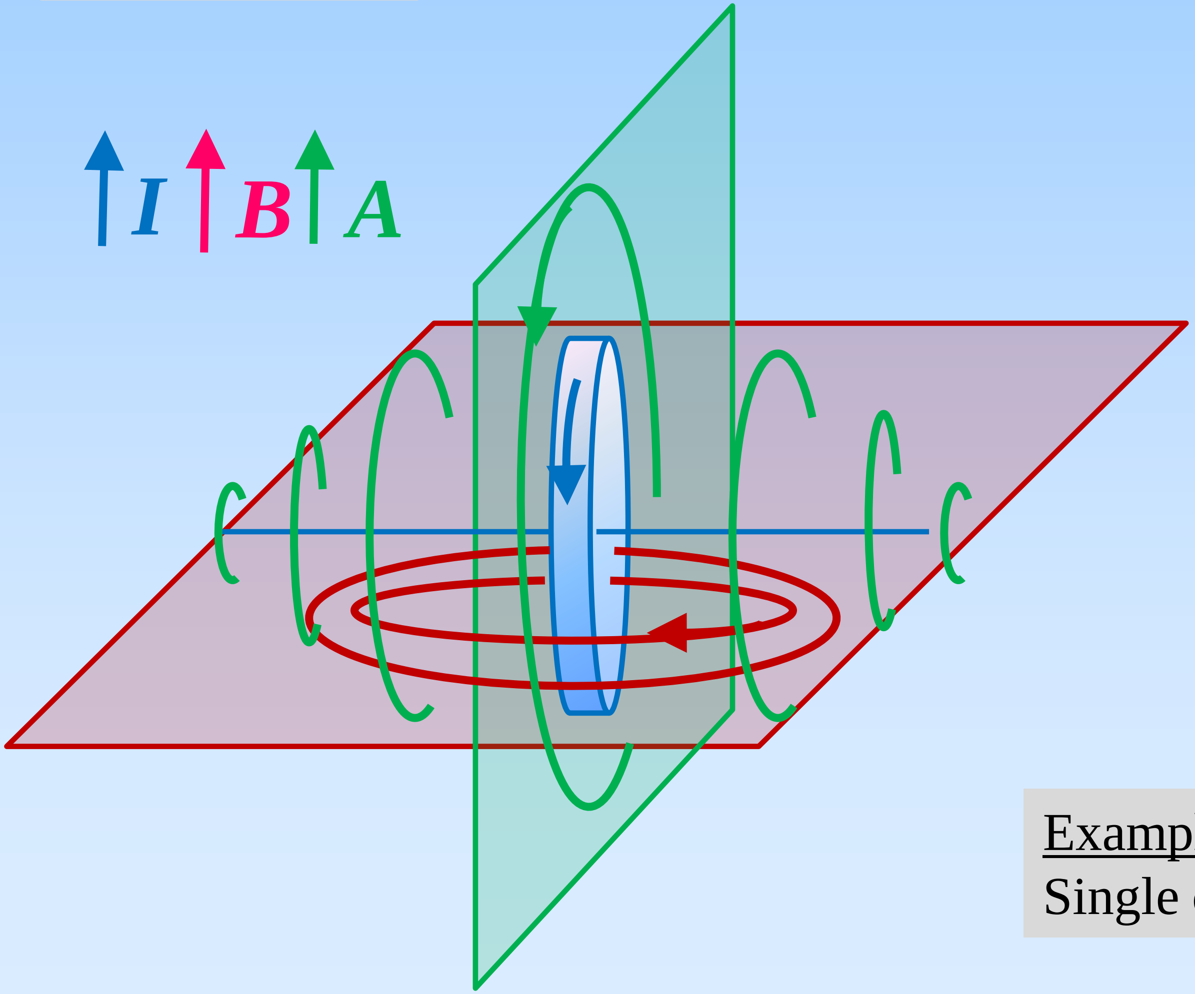


13. Vector potential *A* : examples (4)

general : $\boldsymbol{B} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j} \times \boldsymbol{e}_r}{r^2} dv' \Rightarrow \boldsymbol{A} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j}}{r} dv'$

circuit : $\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_{loop} \frac{\boldsymbol{e}_r}{r^2} dl \Rightarrow \boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint_{loop} \frac{\boldsymbol{dl}}{r}$

Circle circuit



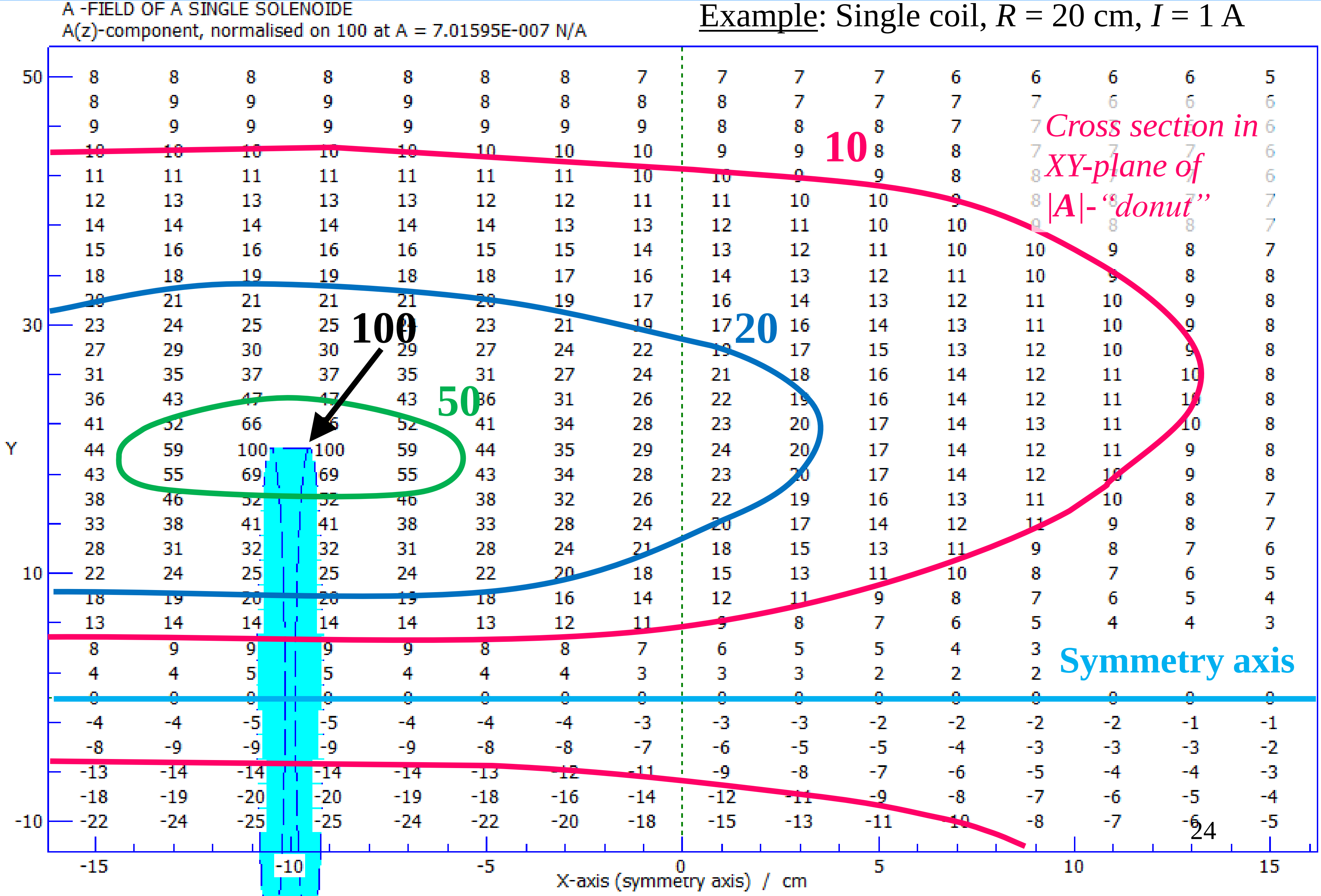
NB. Since *A* is a vector, *A* does not form equipotential planes, as with electric V, with planes ⊥ $\boldsymbol{E}$ $(\boldsymbol{E} = -\nabla V)$

Equi-"modulus" planes of *A* have a donut-shape with the coil as central line.
A is directed tangentially along the donut-surface.

Example:
Single coil, R = 20 cm, I = 1 A

13. Vector potential *A* : examples (4)

<u>Example</u>: Single coil, R = 20 cm, I = 1 A



13. Vector potential *A*

<u>Applications of the vector potential</u>: examples:

- time-dependent fields: $$\boldsymbol{E} = -\nabla \boldsymbol{V} - \frac{\partial \boldsymbol{A}}{\partial t}$$
- electromagnetic waves
- dipolar radiation
- antenna design
- light (and other EM-waves) scattering and transport
    - (e.g. refraction / reflection)

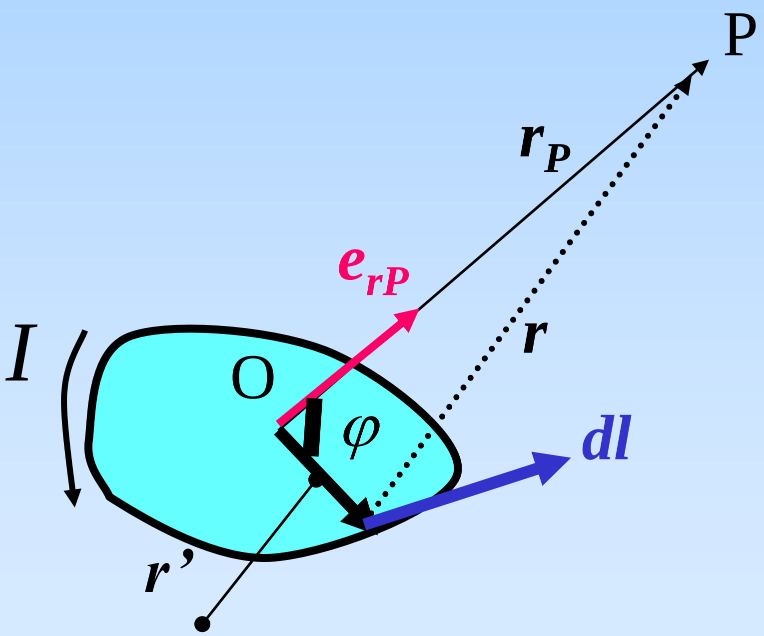
14. Magnetic Dipole (1): Far Field

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j} \times \boldsymbol{e}_r}{r^2} dv' \quad : \text{Biot \& Savart}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\Rightarrow \boldsymbol{A} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\boldsymbol{j}}{r} dv' \quad \text{in general,}$$

$$\text{and } \boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint_{loop} \frac{1}{r} \boldsymbol{dl} \quad \text{for a circuit}$$

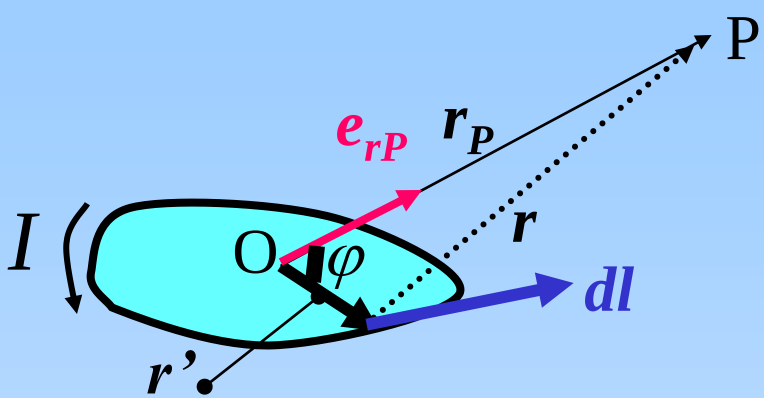


Circuit with current I

Point P outside the circuit

Goal: expression for $\boldsymbol{A}$ in P : $\boldsymbol{A} = \boldsymbol{f}(r_P)$
in stead of $\boldsymbol{A} = \boldsymbol{f}(r')$
for all r'-values in the circuit

14. Magnetic Dipole (1): Far Field



Vector potential : $\boldsymbol{A} = \dfrac{\mu_0 I}{4\pi} \oint \dfrac{\boldsymbol{dl}}{r}$

Goal: expression $\boldsymbol{A} = \boldsymbol{f}\,(r_P)$ in stead of $\boldsymbol{A} = \boldsymbol{f}\,(r')$ for all r'-values in the circuit

$$\frac{1}{r} = \frac{1}{\sqrt{r_P{}^2 + r'^2 - 2 r_P r' . \cos\varphi}}$$

cosine rule

Far field approx.: $r' << r_P$

$$\frac{1}{r} = \frac{1}{r_P \sqrt{1 + \left(\frac{r'}{r_P}\right)^2 - 2\frac{r'}{r_P}\cos\varphi}} \approx \frac{1}{r_P}\left[1 + \frac{r'}{r_P}\cos\varphi + \ldots\text{.higher powers of } \frac{r'}{r_P}\right]$$

(will be neglected)

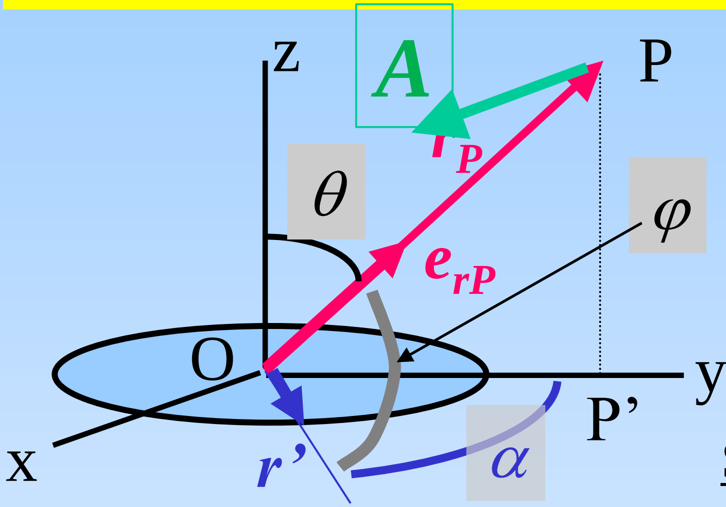
$$\text{Thus : } \quad \boldsymbol{A_P} = \frac{\mu_0 I}{4\pi r_P} \oint \boldsymbol{dl} \quad + \frac{\mu_0 I}{4\pi r_P{}^2} \oint r' . \cos\varphi . \boldsymbol{dl}$$

Monopole-term =0

Dipole-term

14. Magnetic Dipole (2): Dipole Moment

$$\boldsymbol{A_P} = \frac{\mu_0 I}{4\pi r_P{}^2} \oint r'.\cos\varphi.\boldsymbol{dl}$$



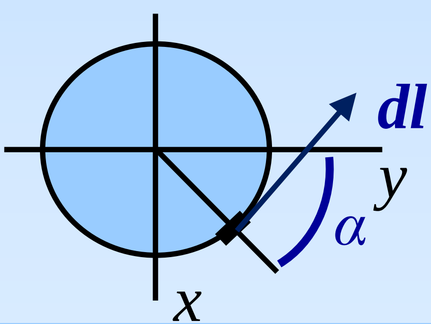
Assume: circular circuit, with radius $R << r_P$

Calculate: $\boldsymbol{A}$ in P in YZ-plane

$$r'.\cos\varphi = \boldsymbol{e_r} \bullet \boldsymbol{r'} = \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} \begin{pmatrix} -R\sin\alpha \\ +R\cos\alpha \\ 0 \end{pmatrix}$$

$$= R\sin\theta\cos\alpha$$

Symmetry: $\boldsymbol{A}$ will have an X-component only, and thus $\boldsymbol{dl}$ as well.



$$dl_x = R.d\alpha.(-\cos\alpha)$$

$$A_x = \frac{\mu_0 I}{4\pi r_P{}^2} \oint (R\sin\theta\cos\alpha)(R.d\alpha)(-\cos\alpha) =$$

$$= \frac{\mu_0 I}{4\pi r_P{}^2} R^2 \sin\theta \oint -\cos^2\alpha.d\alpha \quad = \frac{\mu_0 I}{4\pi r_P{}^2} R^2 \sin\theta.\pi$$

$$A_y = A_z = 0$$

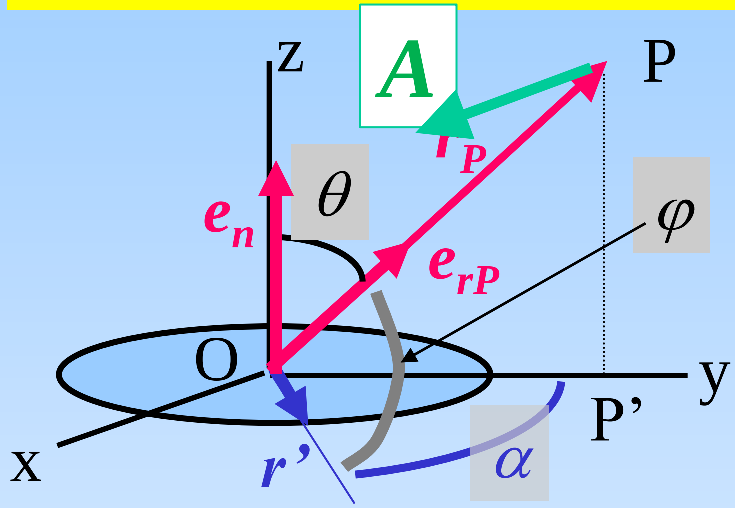
FdM

14. Magnetic Dipole (3): Dipole Moment

$$\boldsymbol{A_P} = \frac{\mu_0 I}{4\pi r_P^{\,2}} \oint r'.\cos\varphi.\boldsymbol{dl}$$

Assume: circular circuit, with radius $R << r_P$

Calculate: $\boldsymbol{A}$ in P in YZ-plane



previous screen:

$$A_x = \frac{\mu_0 I}{4\pi r_P^{\,2}} \oint (R\sin\theta\cos\alpha)(R.d\alpha)(-\cos\alpha) =$$

$$= \frac{\mu_0 I}{4\pi r_P^{\,2}} R^2 \sin\theta \oint -\cos^2\alpha.d\alpha \quad = \frac{\mu_0 I}{4\pi r_P^{\,2}} R^2 \sin\theta.\pi$$

$$A_y = A_z = 0$$

Define: dipole moment:

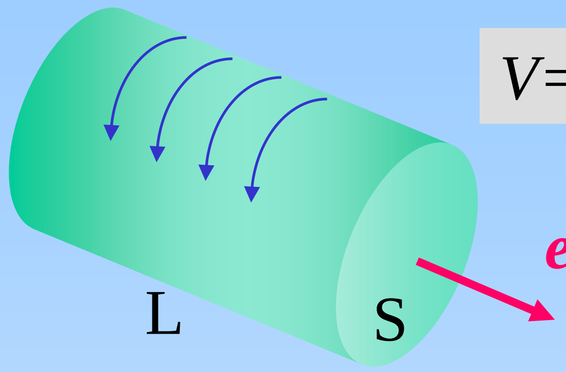
$\boldsymbol{m} = I.\,Area.\,\boldsymbol{e_n} = I\pi R^2.\boldsymbol{e_n}$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi r_P^{\,2}} \boldsymbol{m} \times \boldsymbol{e_r}$$

$\boldsymbol{A} \perp \boldsymbol{e_n}$ and $\boldsymbol{e_r}$; $\boldsymbol{A} // \boldsymbol{e_x}$

15. Magnetization and Polarization

Magnet = set of “elementary circuits” ; n per m^3



$V = SL$

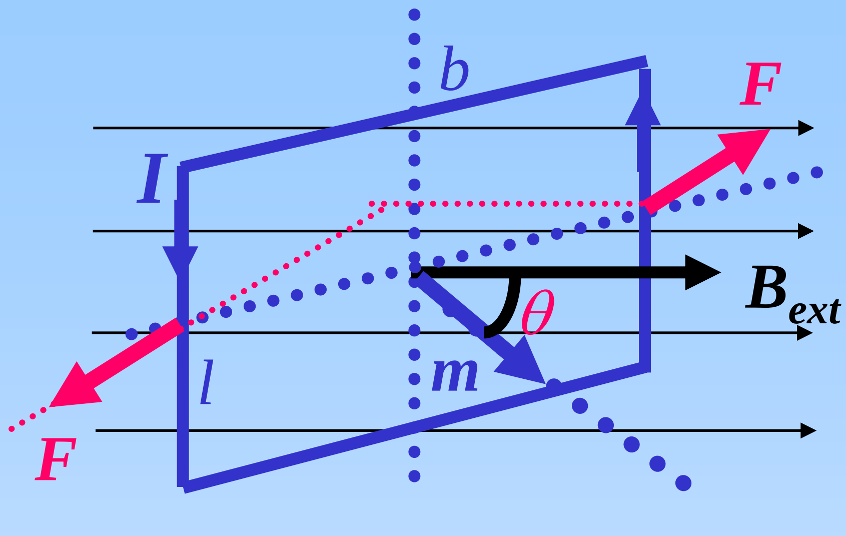
Total surface current = I_{tot}

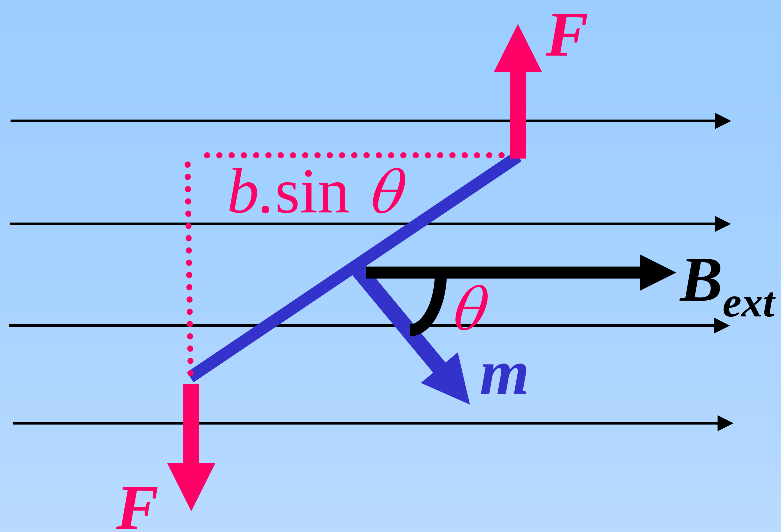
Total magnetic moment = $I_{tot}\ S\ \boldsymbol{e_n}$

Def.: **Magnetization *M* =** magnetic moment / volume = surface current / length

Polarization	Magnetization
Dipole moment: $\boldsymbol{p}$ [Cm]	Dipole moment: $\boldsymbol{m}$ [Am^2]
Polarization $\boldsymbol{P} = n\boldsymbol{p}$ [Cm^{-2}] = surface charge / m^2	Magnetization $\boldsymbol{M} = n\boldsymbol{m}$ [Am^{-1}] = surface current / m

16. Magnetic circuit: Torque and Energy





Torque: moment: $\tau = \; F \,.\, b \sin\,\theta = I\,B_{ext}\, l\, b \sin\,\theta \; = I\,B_{ext}\,S \sin\,\theta$

Magnetic dipole moment: $\boldsymbol{m} = I\,S\,\boldsymbol{e_n}$

Torque: moment: $\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B_{ext}}$

NB. Electric dipole: $\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E_{ext}}$

Potential energy: min for $\theta = 0$; max for $\theta = \pi \;\Rightarrow$

Potential energy: $E_{pot} = -mB_{ext}.\cos\,\theta = -\,\boldsymbol{m} \,.\, \boldsymbol{B_{ext}}$

NB. Electric dipole: $E_{pot} = -\,\boldsymbol{p} \,.\, \boldsymbol{E_{ext}}$

17. Electret and Magnet

<u>Electret</u>

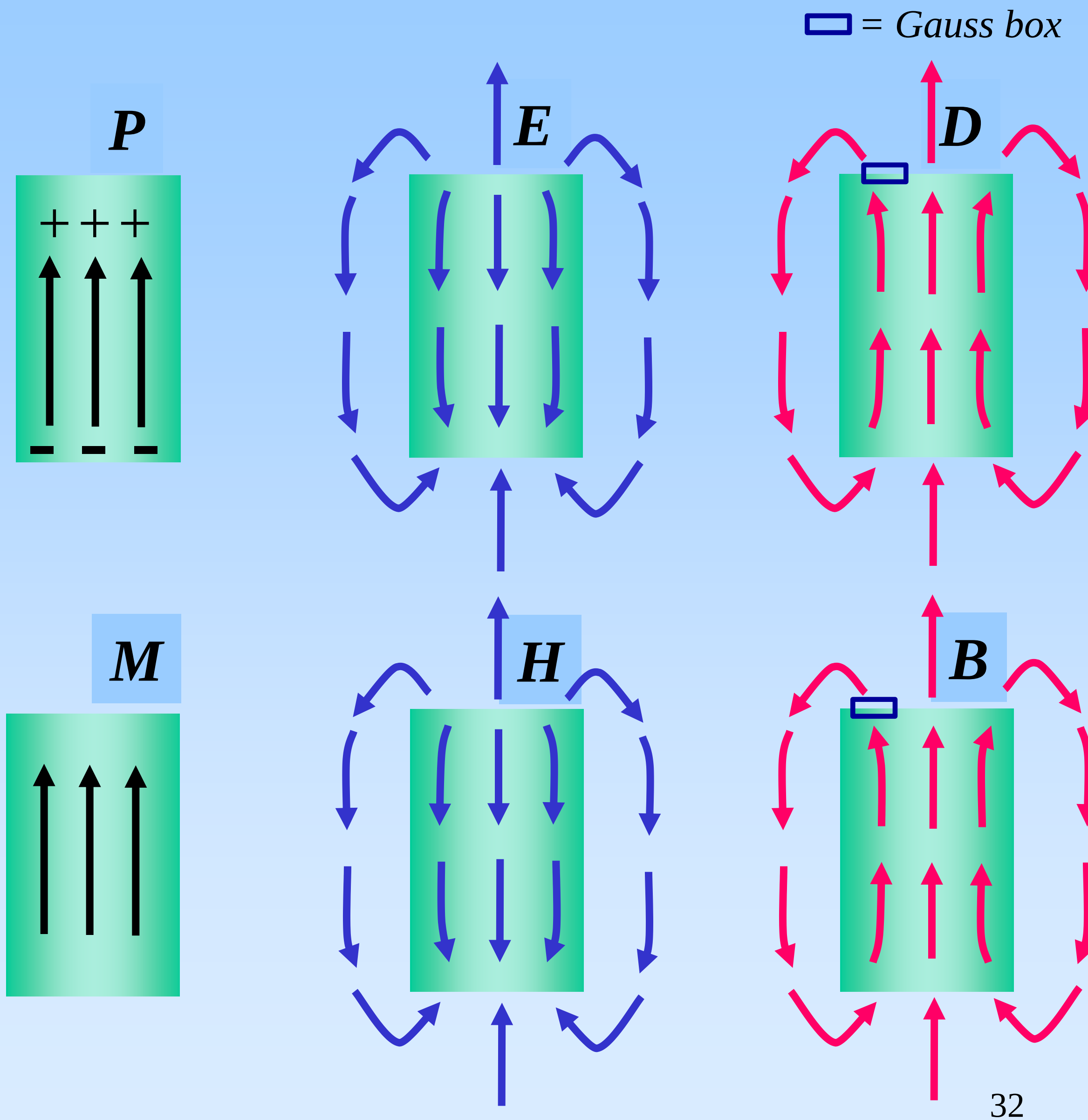
$$\boldsymbol{E} = (\boldsymbol{D}-\boldsymbol{P})/\ \varepsilon_0$$
$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

$$\oint \boldsymbol{E} \bullet \boldsymbol{dl} = 0$$
$$\oiint \boldsymbol{D} \bullet \boldsymbol{dS} = Q_f = 0$$

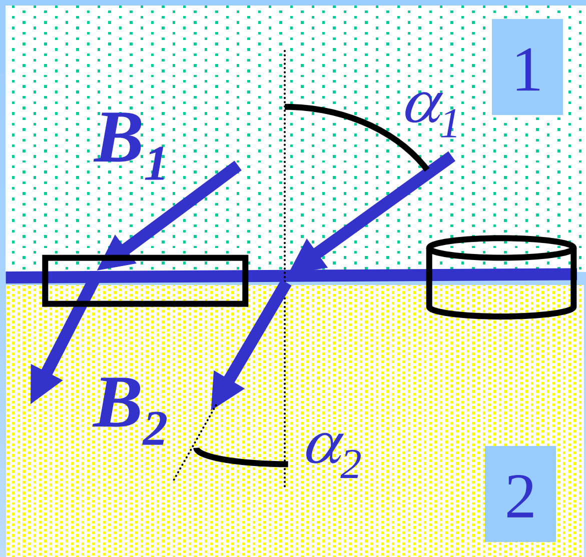
<u>Magnet</u>

$$\boldsymbol{H} = \boldsymbol{B}/\ \mu_0 - \boldsymbol{M}$$
$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}$$

$$\oint \boldsymbol{H} \bullet \boldsymbol{dl} = I_f = 0$$
$$\oiint \boldsymbol{B} \bullet \boldsymbol{dS} = 0$$



18. ***B***- and ***H***-fields at interface



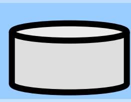
<u>Given</u>: $\boldsymbol{B_1}$; μ_1 ; μ_2

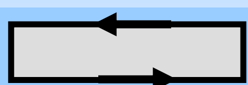
<u>Question</u>: Calculate $\boldsymbol{B_2}$

<u>Needed:</u> "Interface-crossing relations":

$$\oint \boldsymbol{H} \bullet \boldsymbol{dl} = I_{free} \quad \text{and} \quad \oiint \boldsymbol{B} \bullet \boldsymbol{dA} = 0$$

<u>Relation ***H*** and ***B*:**</u> $\boldsymbol{B} = \mu_0 \, \mu_r \, \boldsymbol{H}$

Gauss box  : $B_1.A\cos\alpha_1 - B_2.A\cos\alpha_2 = 0 \;\Rightarrow\; B_{1\perp} = B_{2\perp}$

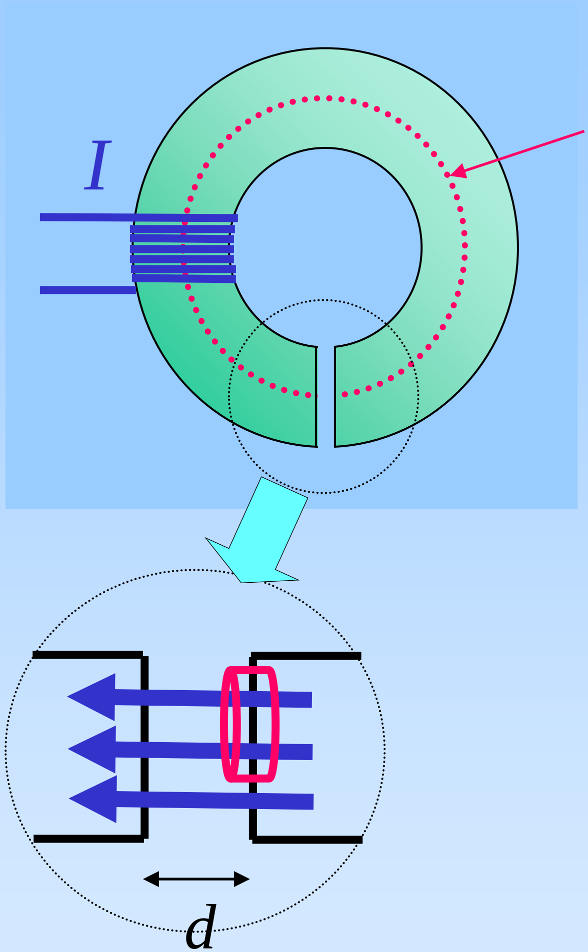
Circuit:  : no I : $H_1.L\sin\alpha_1 - H_2.L\sin\alpha_2 = 0 \;\Rightarrow\; H_{1//} = H_{2//}$

$$\frac{B_1}{H_1}\frac{1}{\tan\alpha_1}\frac{A}{L} = \frac{B_2}{H_2}\frac{1}{\tan\alpha_2}\frac{A}{L}$$

$$\frac{\mu_{r1}}{\mu_{r2}} = \frac{\tan\alpha_1}{\tan\alpha_2}$$

NB. For dielectric materials: $\dfrac{\varepsilon_{r1}}{\varepsilon_{r2}} = \dfrac{\tan\alpha_1}{\tan\alpha_2}$

19. Toroid with air gap



Core line; radius R length L

Suppose:

- toroid solenoid: R, L, N, μ_r ;
- air gap, $\mu_r = 1$, width $d << R$;
  g = gap ; m = metal

Question: Determine H_g in gap

Relations: $\oint \boldsymbol{H} \bullet \boldsymbol{dl} = I_f$ $\quad$ $\boldsymbol{B} = \mu_0 \mu_r \boldsymbol{H}$

$$\oiint \boldsymbol{B} \bullet \boldsymbol{dA} = 0$$

$$H_g d + H_m (L-d) = N I$$

$$B_g = B_m$$

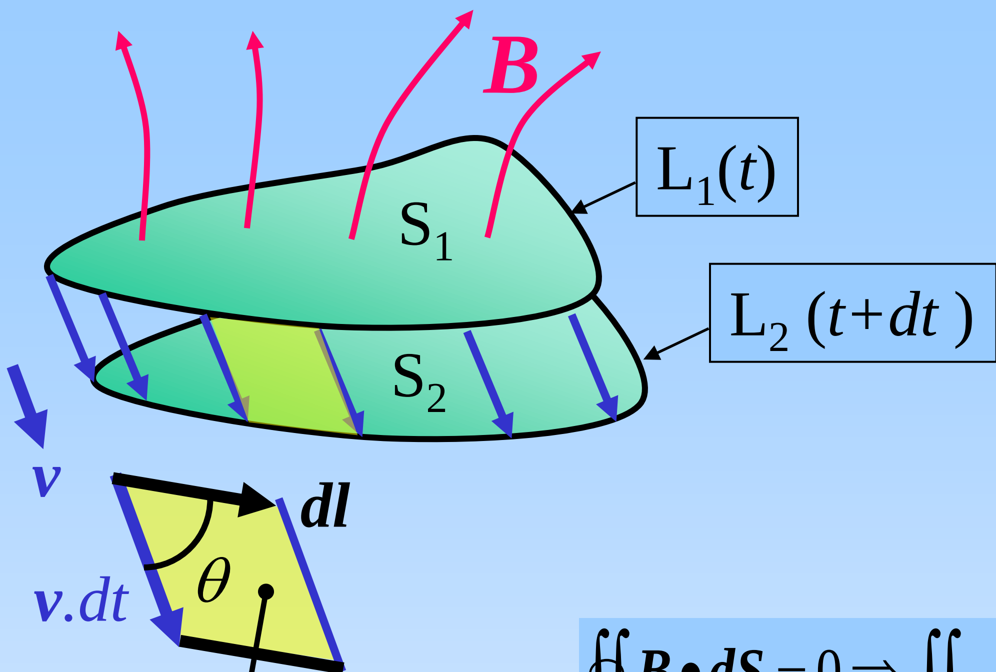
$$B_g = \mu_o H_g \quad ; \quad B_m = \mu_0 \mu_r H_m$$

Result: $$H_g = \frac{\mu_r N I}{L + d(\mu_r - 1)} \quad \text{and} \quad H_m = \frac{H_g}{\mu_r}$$

This is the technical magnetic analogon of the homogeneous electric field in an ideal capacitor

NB. In gap: $H_g \sim \mu_r N$.

20. Induction: conductor moves in field



Area = $v\ dt\ dl\ \sin\theta$

<u>Suppose:</u> circuit L moves with velocity $\boldsymbol{v}$ through field $\boldsymbol{B}$

<u>Question:</u> Show equivalence:

$$V_{ind} = \oint_L \boldsymbol{E} \bullet \boldsymbol{dl} = -\frac{d}{dt}\iint_S \boldsymbol{B} \bullet \boldsymbol{dS}$$

<u>Gauss box</u>: top lid S_1 in L_1 at t, bottom lid S_2 in L_2 at $t+dt$

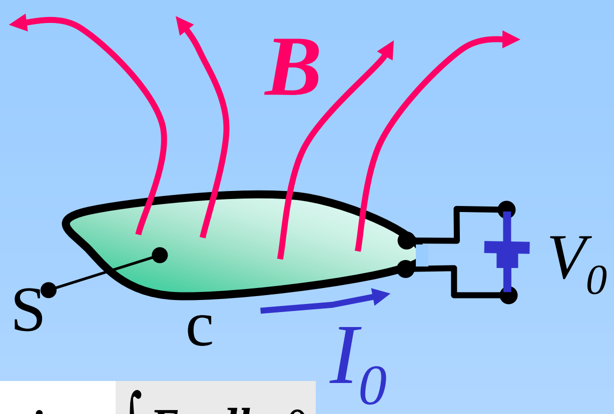
$$\oint\!\!\oint \boldsymbol{B} \bullet \boldsymbol{dS} = 0 \Rightarrow \iint_{wall} \ = -\iint_{top} \ - \iint_{bottom} \ = -\Phi(t) + \Phi(t+dt) = d\Phi(t)$$

$$d\Phi(t) = \iint_{wall} \ = \iint_{wall} \boldsymbol{B} \bullet \boldsymbol{e_n}[v.dt.dl.\sin\theta] = \oint_L \boldsymbol{B} \bullet (\boldsymbol{v}.dt \times \boldsymbol{dl}) = dt.\oint_L -(\boldsymbol{v} \times \boldsymbol{B}) \bullet \boldsymbol{dl}$$

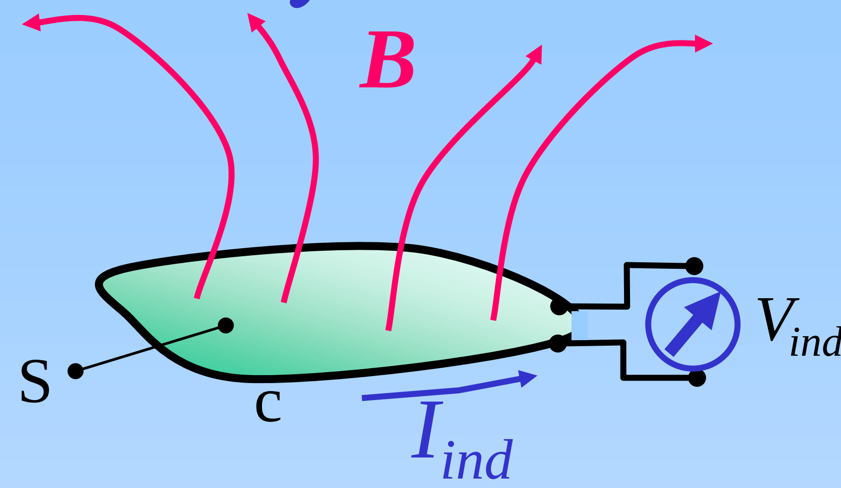
$$\Rightarrow \frac{d\Phi(t)}{dt} = \oint_L -(\boldsymbol{v} \times \boldsymbol{B}) \bullet \boldsymbol{dl} = +\oint_L \boldsymbol{E_n} \bullet \boldsymbol{dl}$$

$\boldsymbol{E_n}$ = non-electrostatic field;
Circuit closed by load
(e.g. voltmeter or resistance)

21. Induction: Faraday's Law



Static: $\oint_c \boldsymbol{E} \bullet \boldsymbol{dl} = 0$

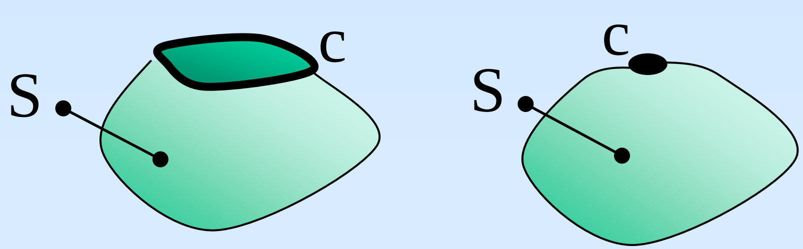


Dynamic: Suppose Φ changes ⇒

$V_{ind} \neq 0 \Rightarrow$
Non-electrostatic field $\boldsymbol{E_N}$

$$V_{ind} = \oint_c \boldsymbol{E}_N \bullet \boldsymbol{dl} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\iint_S \boldsymbol{B} \bullet \boldsymbol{dS}$$

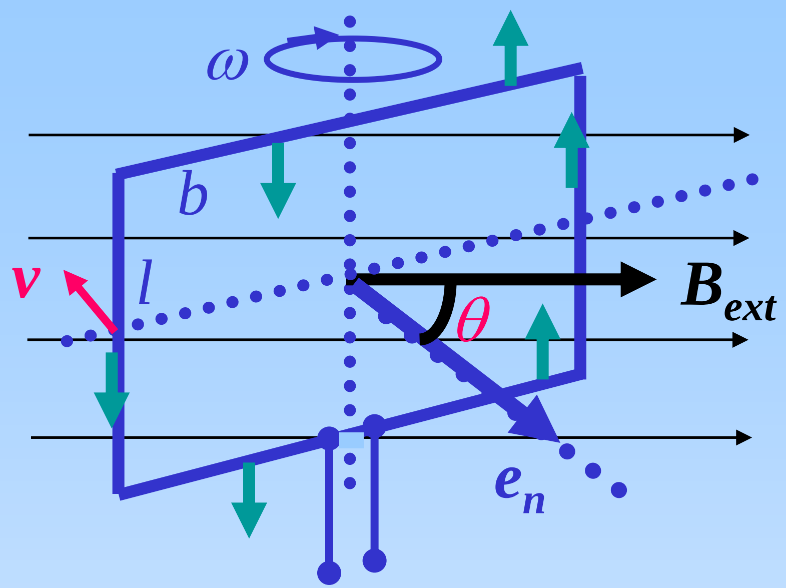
Consequence: Let circuit c shrink, while keeping S constant.

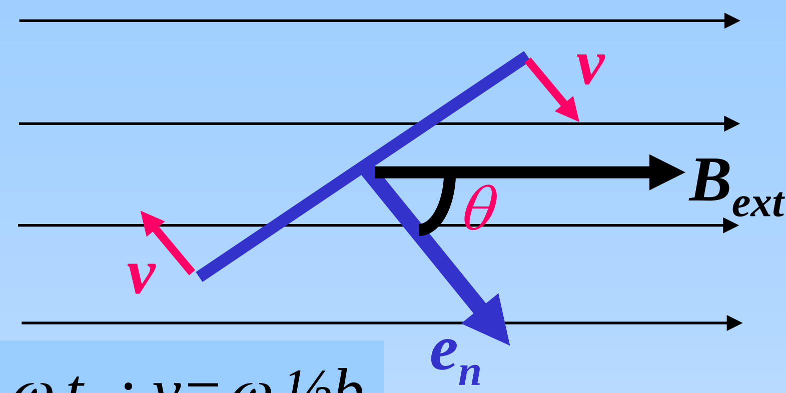


For closed surface :

$$\oint_c \boldsymbol{E}_N \bullet \boldsymbol{dl} = 0 \Rightarrow \oiint_S \boldsymbol{B} \bullet \boldsymbol{dS} = 0$$

22. Induction in rotating circuit frame





$\theta = \omega t \;\; ; \; v = \omega \tfrac{1}{2} b$

<u>Induction potential difference</u> :

(I) Using Lorentz force:

↓ $= \boldsymbol{E}_N$

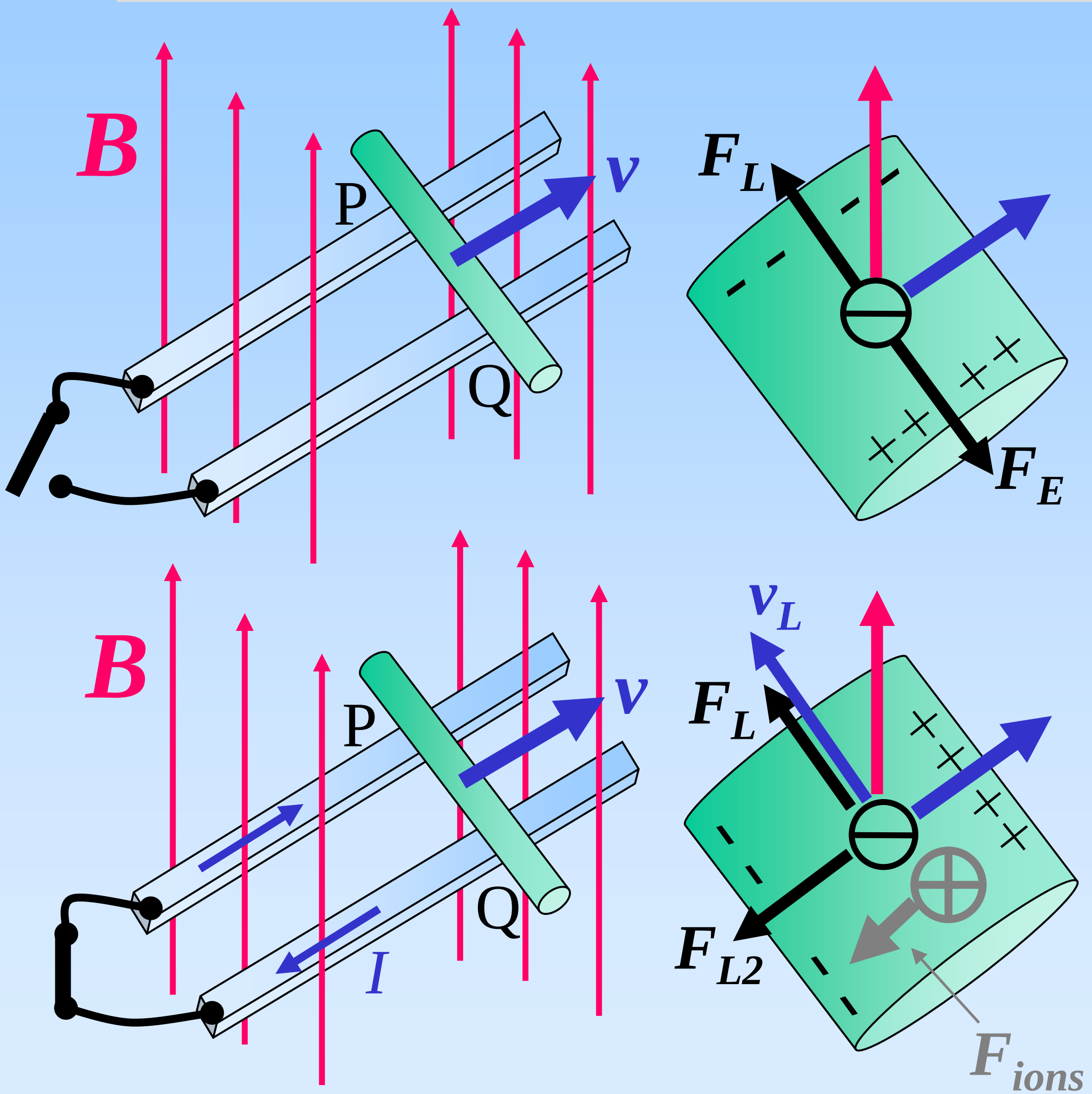
$$V_{ind} = \oint \boldsymbol{E}_N \bullet \boldsymbol{dl} = \oint (\boldsymbol{v} \times \boldsymbol{B}_{ext}) \bullet \boldsymbol{dl} =$$
$$2lvB_{ext} \sin\theta = 2lvB_{ext} \sin\omega t$$

(II) Using flux change:

$$V_{ind} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \iint \boldsymbol{B}_{ext} \bullet \boldsymbol{dS} = -B_{ext} \frac{d}{dt}[lb\cos\theta] =$$
$$B_{ext}\, lb\omega \sin\omega t = 2B_{ext}\, lv \sin\omega t$$

23. Electromagnetic brakes

Why is a conducting wire decelerated by a magnetic field?



Case I: switch open

Electrons feel F_L

Potential difference V_{PQ} : P - , Q + (= Hall effect)

F_L counteracted by F_E

until $F_L = F_E$

Case II: switch closed

F_L moves electrons: v_L,

which causes I and F_{L2} ,

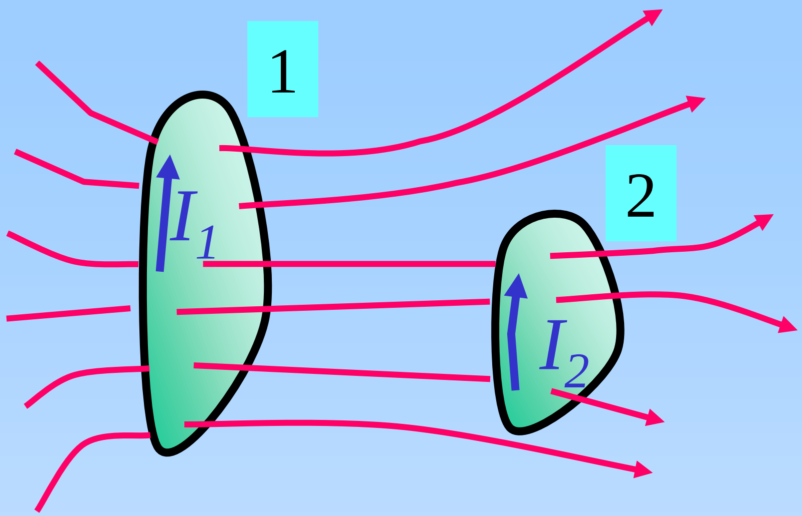
which causes electric field,

which will act on positive metal ions: F_{ions}. →

brake on → deceleration

FdM

24. Coupled circuits (1): *M* and *L*



Suppose: circuit 1 with current I_1

Part of flux from 1 will pass through 2 : Φ_{21}

$\Phi_{21} \sim I_1$

Definition: $\Phi_{21} = M_{21}.I_1$

M : coefficient of mutual induction : "***Mutual Inductance***" :

$[\,H\,] = [\,NA^{-1}m^{-1}.m^2.A^{-1}] = [\,NmA^{-2}\,]$

Suppose: Circuit 2 has current as well: I_2

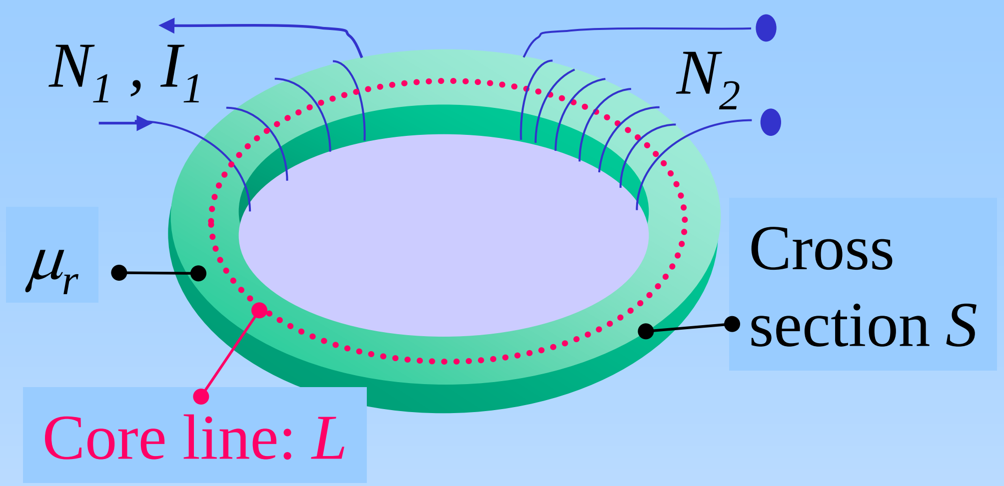
Flux through 2: $\Phi_2 = \Phi_{21} + \Phi_{22} = M_{21}\, I_1 + M_{22}\, I_2$

$M_{22} = L_2$

L : coefficient of self-induction "***(Self) Inductance***":

M and *L* are geometrical functions (shape, orientation, distance etc.)

24. Coupled circuits (2): toroid



Question: determine induction coefficient of 1 in 2: mutual induction M_{21}

Flux from 1 through S : $\Phi_S = BS = \mu_0 \mu_r N_1 I_1 S / L$

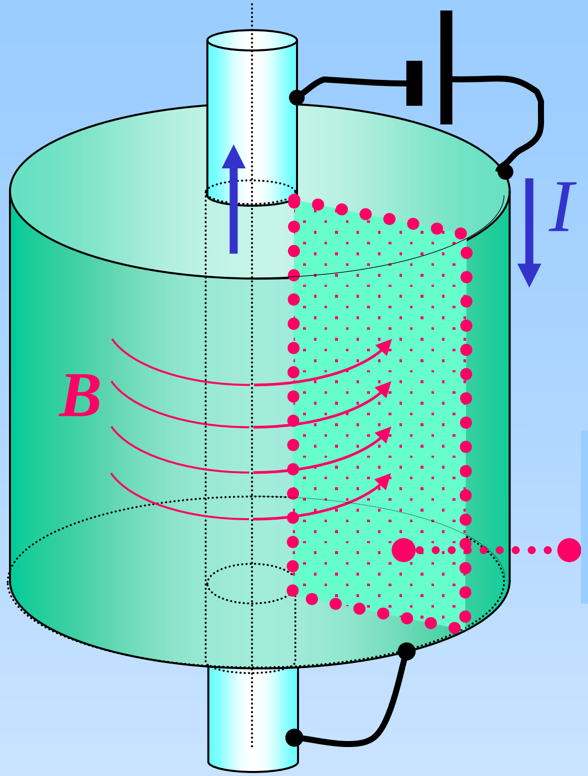
Linked flux from 1 through 2: $\Phi_{21} = N_2 \Phi_S = \mu_0 \mu_r N_1 N_2 I_1 S / L$

Coefficient of mutual induction: $M_{21} = \mu_0 \mu_r N_1 N_2 S / L$

This expression is symmetrical in 1 and 2: $M_{21} = M_{12}$

This result is generally valid: $M_{ij} = M_{ji}$

25. Coax cable : Self inductance



Radii: *a* and *b* (*a*<*b*)
Length: *l*
Current: *I* ; choice: inside = upward

Ampere:
B-field tangential

$$B(r) = \frac{\mu_0 \mu_r I}{2\pi r}$$

Flux through circuit:

$$\Phi_B = \iint_S \boldsymbol{B} \bullet \boldsymbol{dS} = \int_0^l dl \int_a^b \frac{\mu_0 \mu_r I}{2\pi r}\, dr = \frac{\mu_0 \mu_r I . l}{2\pi} \ln \frac{b}{a}$$

Self-inductance (coefficient of self-induction): per unit of length:

$$L = \frac{\mu_0 \mu_r}{2\pi} \ln \frac{b}{a}$$

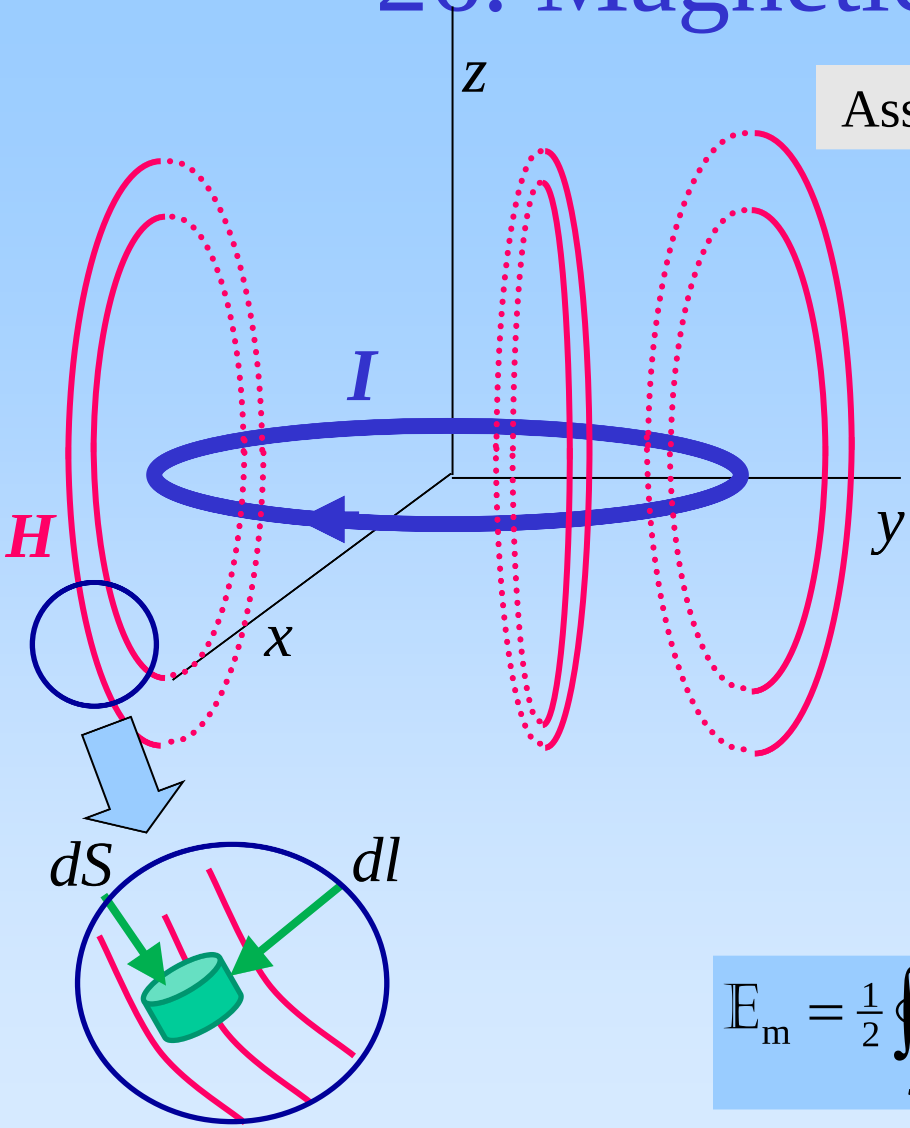
(Compare with capacity of coax cable, per meter:

$$C = 2\pi \varepsilon_0 \varepsilon_r \left[\ln \frac{b}{a} \right]^{-1}$$

NB. In oscillator circuits (*L* and *C* in series): frequency ω: $\omega^{-2} = LC = \varepsilon_0\, \varepsilon_r\, \mu_0\, \mu_r\, l^2$

FdM

26. Magnetic Field Energy



Assume: currents may be everywhere in space.

Example: 1 circuit c in XY-plane

***H (B)*-field**: closed curves ⊥ circuit.

Magnetic power: $d\mathbb{E}_m/dt =$

$= V_{ind}\, I = (d\Phi/dt).\, I = L\,(dI/dt).\, I$

Magn. Energy: $\mathbb{E}_m = \frac{1}{2}\, L.I^2 = \frac{1}{2}\,\Phi\, I$

Space = volume of "tiny" Gauss boxes, with normal // ***H*** (or ***B***).

$$\Phi = \oiint_S \boldsymbol{B} \bullet \boldsymbol{dS} \;\; ; \;\; I = \oint \boldsymbol{H} \bullet \boldsymbol{dl}$$

$$\mathbb{E}_m = \tfrac{1}{2} \oiint_S \boldsymbol{B} \bullet \boldsymbol{dS} \oint_c \boldsymbol{H} \bullet \boldsymbol{dl} = \tfrac{1}{2} \iiint_V \boldsymbol{B} \bullet \boldsymbol{H}\; .\boldsymbol{dS} \bullet \boldsymbol{dl} =$$

$$= \tfrac{1}{2} \iiint_V \boldsymbol{B} \bullet \boldsymbol{H}\; dv$$

Compare: electric energy:

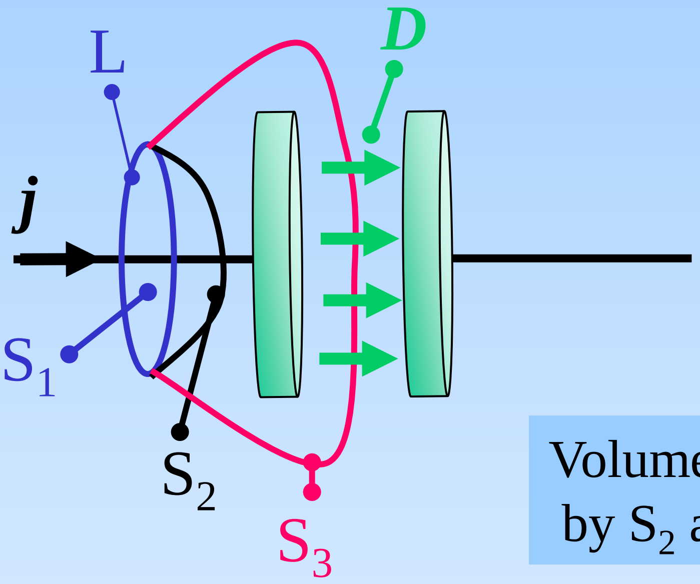
$$\mathbb{E}_{\mathbb{E}} = \tfrac{1}{2} \iiint_v \boldsymbol{D} \bullet \boldsymbol{E}\; dv$$

27. Maxwell's Fix of Ampere's Law

Induction : $\oint_c \boldsymbol{E}_n \bullet \boldsymbol{dl} = -\frac{d}{dt}\iint_S \boldsymbol{B} \bullet \boldsymbol{dS}$

Changing ***B***-field causes an ***E***-field

<u>Question</u>: Does a changing ***E***-field cause a ***B***-field ?



<u>Suppose</u>: chargeing a capacitor using ***j***

L encloses S_1 : $\oint_L \boldsymbol{B} \bullet \boldsymbol{dl} = \mu_0 \iint_{S_1} \boldsymbol{j} \bullet \boldsymbol{dS}$

L encloses S_2 : $\oint_L \boldsymbol{B} \bullet \boldsymbol{dl} = \mu_0 \iint_{S_2} \boldsymbol{j} \bullet \boldsymbol{dS}$

Volume enclosed by S_2 and S_3: $-\oiint_{S_2} \boldsymbol{j} \bullet \boldsymbol{dS} = -\frac{dQ_f}{dt} = -\frac{d}{dt}\iint_{S_3} \boldsymbol{D} \bullet \boldsymbol{dS}$

L encloses S_3 : $\oint_L \boldsymbol{B} \bullet \boldsymbol{dl} = \mu_0 \frac{d}{dt}\iint_{S_3} \boldsymbol{D} \bullet \boldsymbol{dS}$

In general : $\oint_L \boldsymbol{B} \bullet \boldsymbol{dl} = \mu \iint_S \boldsymbol{j} \bullet \boldsymbol{dS} + \mu \frac{d}{dt}\iint_S \boldsymbol{D} \bullet \boldsymbol{dS}$

27. Maxwell's Fix of Ampere's Law

$$\text{Induction}: \quad \oint_c \boldsymbol{E}_n \bullet \boldsymbol{dl} = -\frac{d}{dt}\iint_S \boldsymbol{B} \bullet \boldsymbol{dS}$$

$$\text{Maxwell}: \oint_L \boldsymbol{B} \bullet \boldsymbol{dl} = \mu \iint_S \boldsymbol{j} \bullet \boldsymbol{dS} + \mu \frac{d}{dt}\iint_S \boldsymbol{D} \bullet \boldsymbol{dS}$$

A changing ***B***-field causes an ***E***-field around itself

A changing ***E***-field causes a ***B***-field around itself

This is the starting point for self-propagating

ELECTROMAGNETIC WAVES

(see presentation about Waves)