University of Twente
Department Applied Physics

First-year course on

Electromagnetism

Electricity: Topics

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Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

Electromagnetism

Part I: Electricity - topics

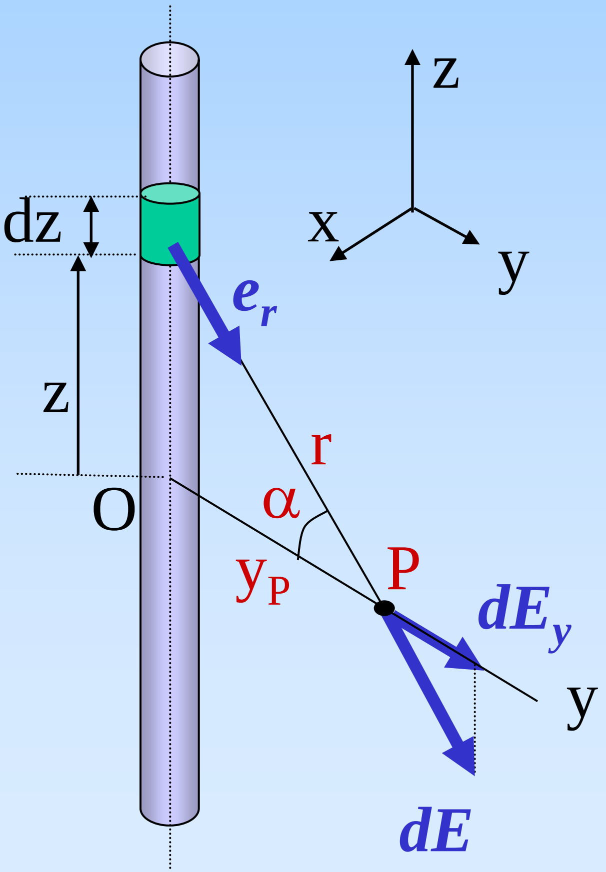
Contents:

1-13: Movie Part A - 14-32 : Part B

1. *E*-field of a long wire

Analysis:

- ∞ long wire: λ [C/m]
- cylindrical symmetry



Problem: $\boldsymbol{E_P}$ in point $P(0,y_P,0)$

Approach:

$$\boldsymbol{dE} = \frac{dQ}{4\pi\varepsilon_0 r^2}\boldsymbol{e_r}$$

- Charge element: dQ=λ.dz
- Symmetry ⇒ y-component only !!

$\boldsymbol{e_r} \cdot \boldsymbol{e_y} = \cos\alpha$

$$\boldsymbol{E} = \int \boldsymbol{dE_y} = \int_{-\infty}^{\infty} \frac{\lambda . dz}{4\pi\varepsilon_0 r^2}\cos\alpha = \frac{\lambda}{2\pi\varepsilon_0 y_P}$$

r, $\boldsymbol{e_r}$ and α are *f* (z) :

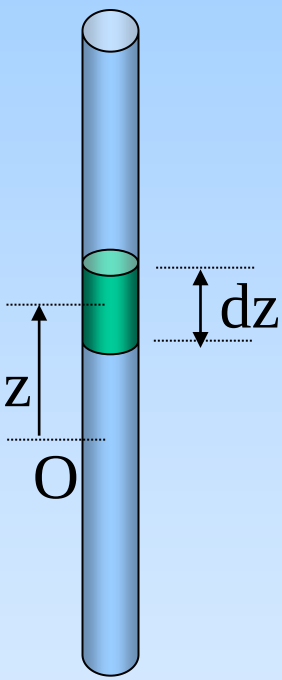
$$r = \sqrt{y_P^{\,2} + z^2}\ ; \cos\alpha = \frac{y_P}{r}$$

Conclusion: ***E*** radial symmetry

2. Charge elements (1): rod, ring, disk

Line

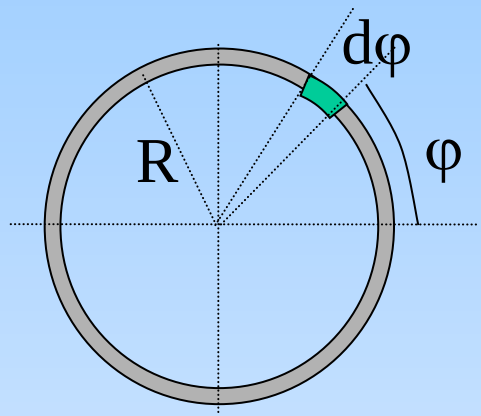
λ [C/m]



$dQ = \lambda . dz$

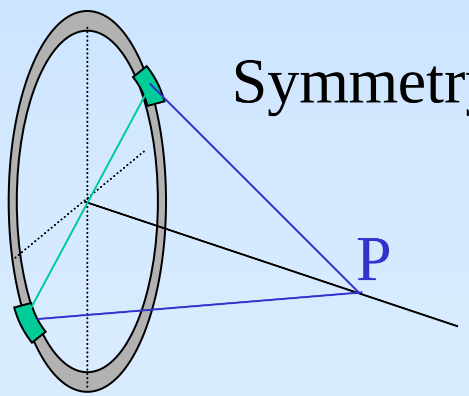
Ring , radius R

λ [C/m]



$dQ = \lambda . R . d\varphi$

Symmetry:

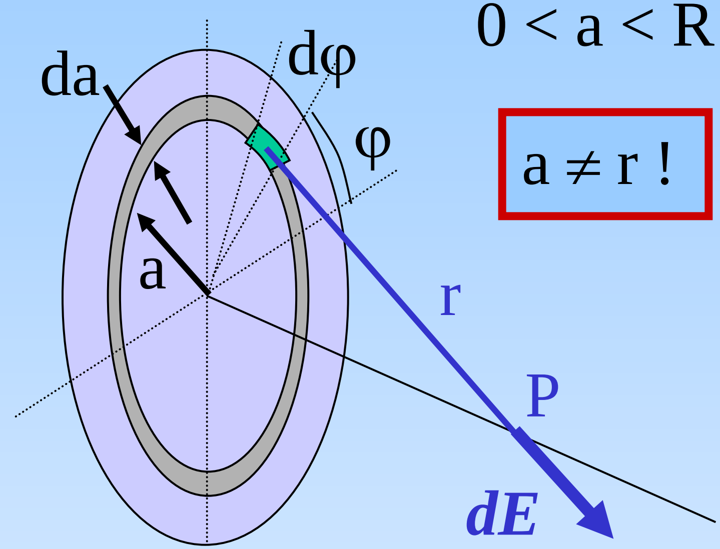


Disk , radius R

σ [C/m²]

$0 < a < R$

$a \neq r$!

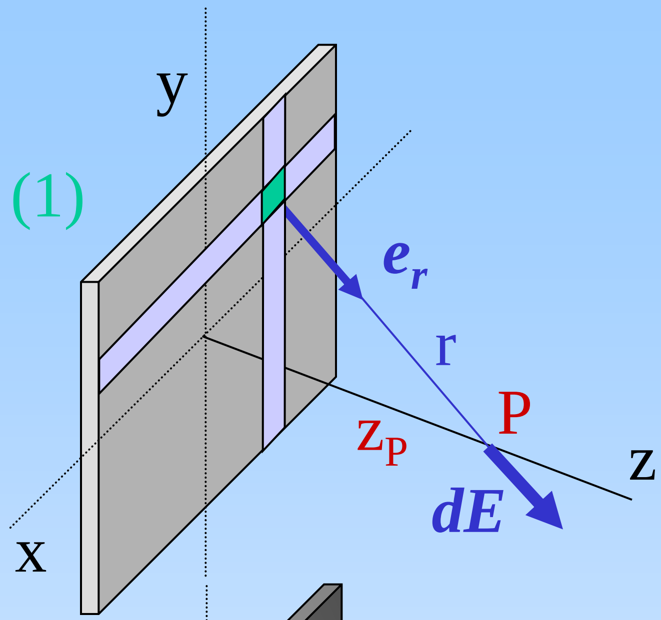


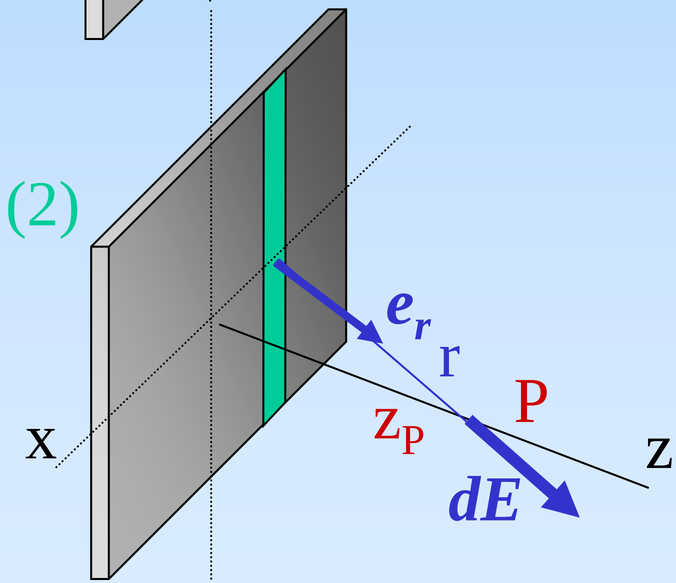
$dQ = \sigma . dA = \sigma\,(a . d\varphi)\,da$

If σ homogeneous:

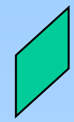
$dQ = \sigma\,.2\pi a . da$

2. Charge elements (2): thin plate





Thin plate , σ [C/m^2]

(1)  = dA = dx.dy , at (x,y)

dQ = σ dA = σ dx.dy

$\boldsymbol{dE} = dE_x\, \boldsymbol{e_x} + dE_y\, \boldsymbol{e_y} + dE_z\, \boldsymbol{e_z}$

if plate ∞ large : $\boldsymbol{dE} \,//\, \boldsymbol{e_z}$

(2) if $\sigma = f\,(x)$ only:

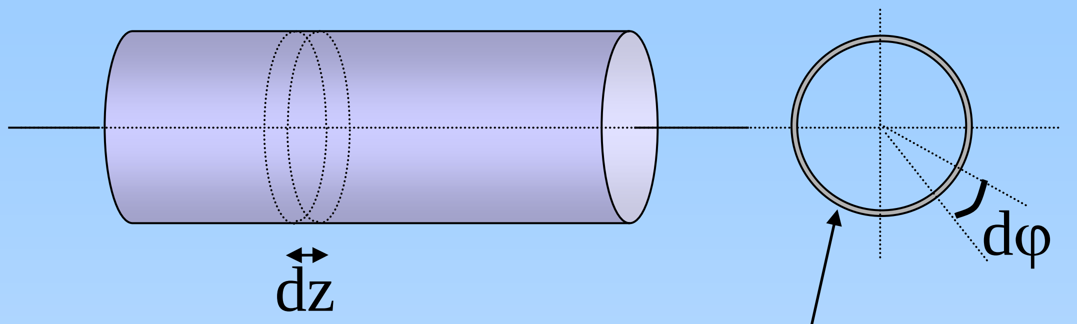
Use result for ∞ long wire:

$$\mathbf{d}E_z = \frac{\mathbf{d}\lambda}{2\pi\varepsilon_0 z_P}$$

dE in XZ-plane

with dλ = σ. dx

2. Charge elements (3): tube and rod



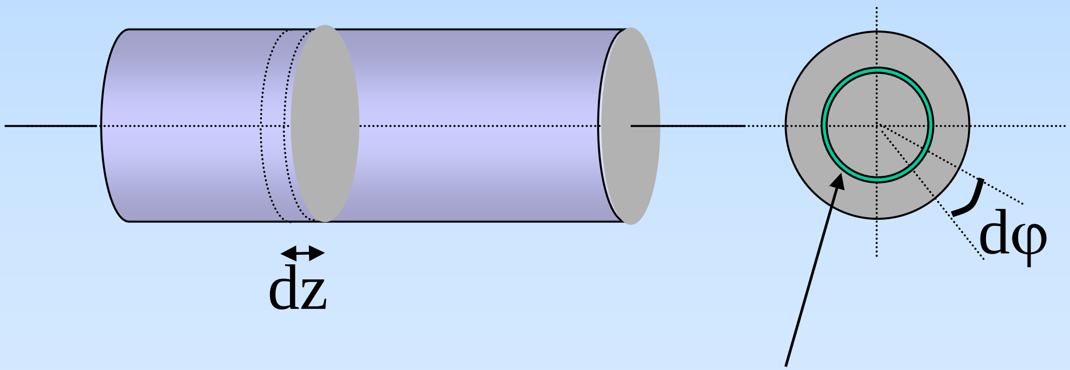
Cross section: thin ring, radius R

Tube , radius R

σ [C/m^2]

$dA = R.\ d\varphi.\ dz$

$dQ = \sigma\ R.\ d\varphi.\ dz$



Cross section: ring, radius a, width da

Avoid using r for radius !!

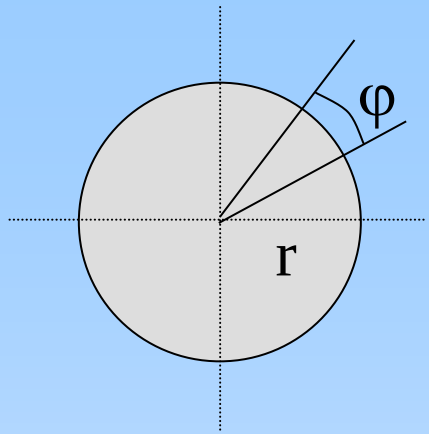
Solid rod , radius R

ρ [C/m^3]

$dV = (a.d\varphi).da.dz$

$dQ = \rho\ (a.d\varphi).da.dz$

3. Arc angle and Solid angle

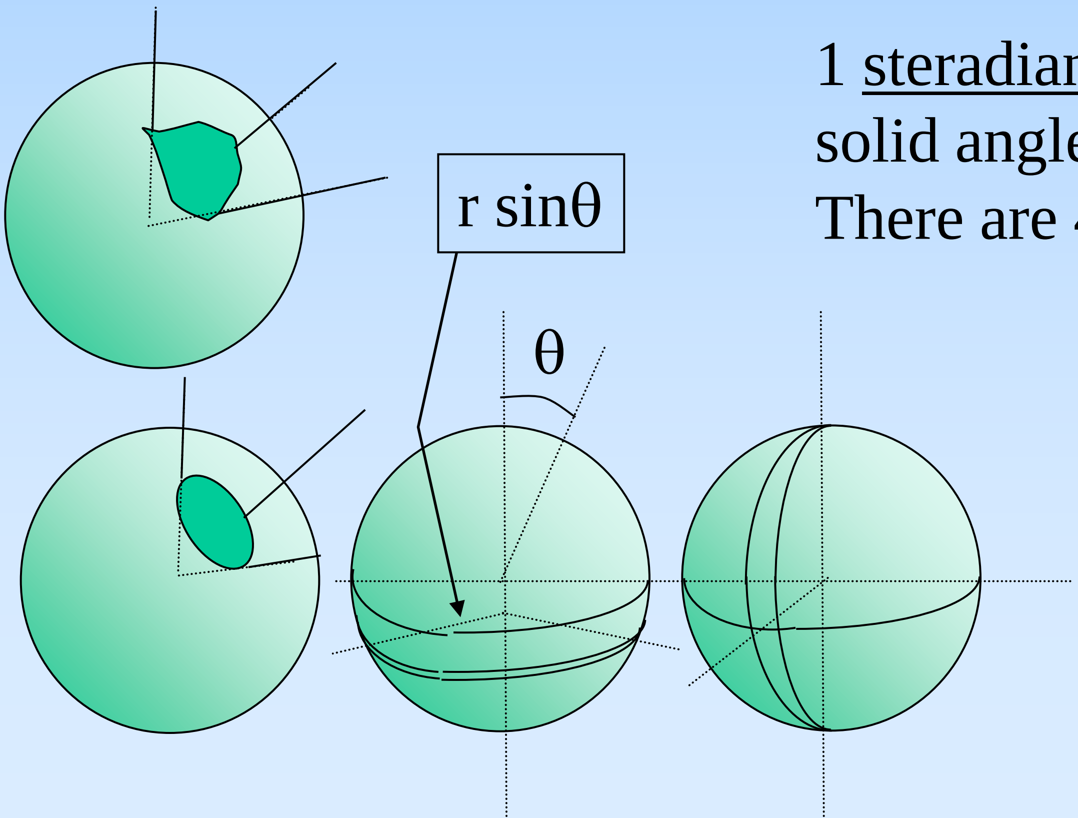


1 <u>radian</u> [rad] ≡

angle with arc length = radius r.

2π radians on circumference.

1 <u>steradian</u> [sr] ≡
solid angle with surface area = r^2.
There are 4π sr on the surface.

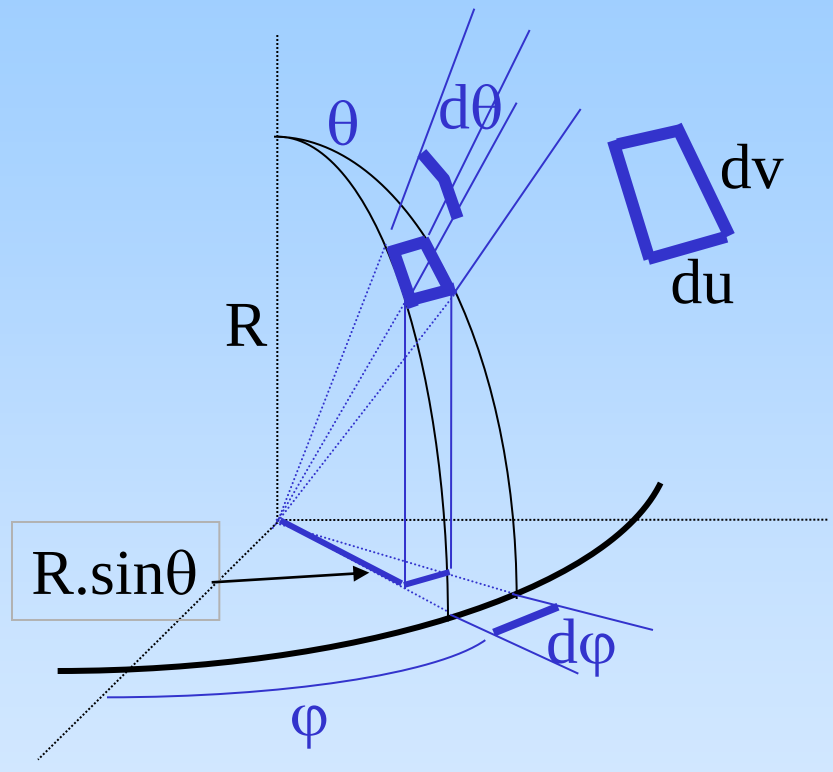


Solid angle:

$d\Omega = (\sin\theta . d\varphi)\, d\theta$

4. Charge elements (4): on/in a sphere

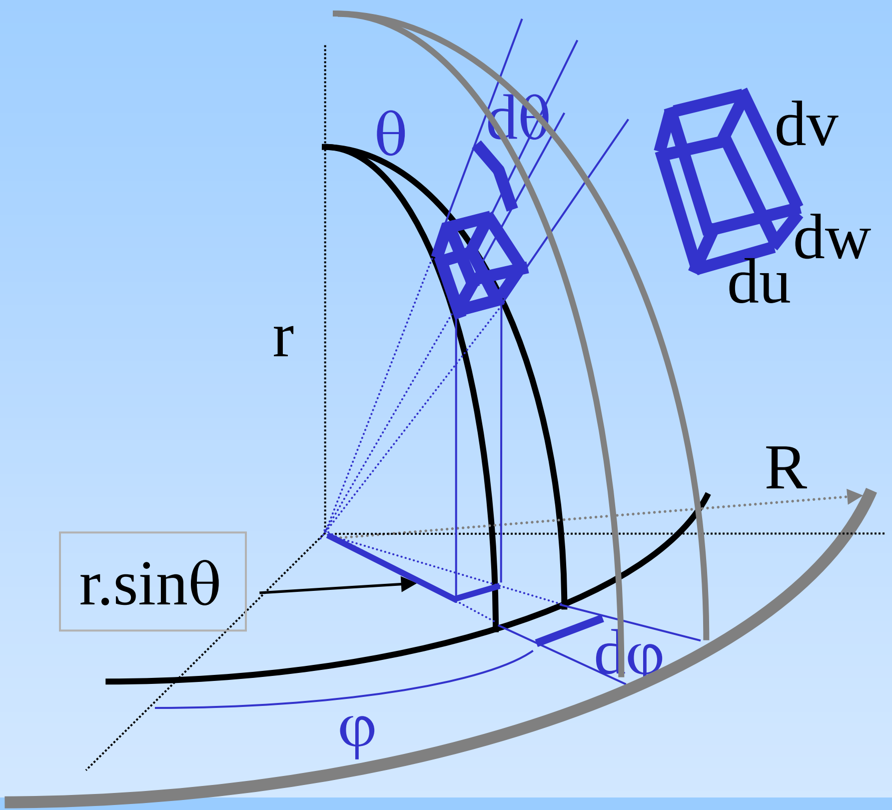
Spherical surface element



dA = du.dv = (R.sinθ.dφ).(Rdθ)

0 < φ < 2π ; 0 < θ < π

Spherical volume element



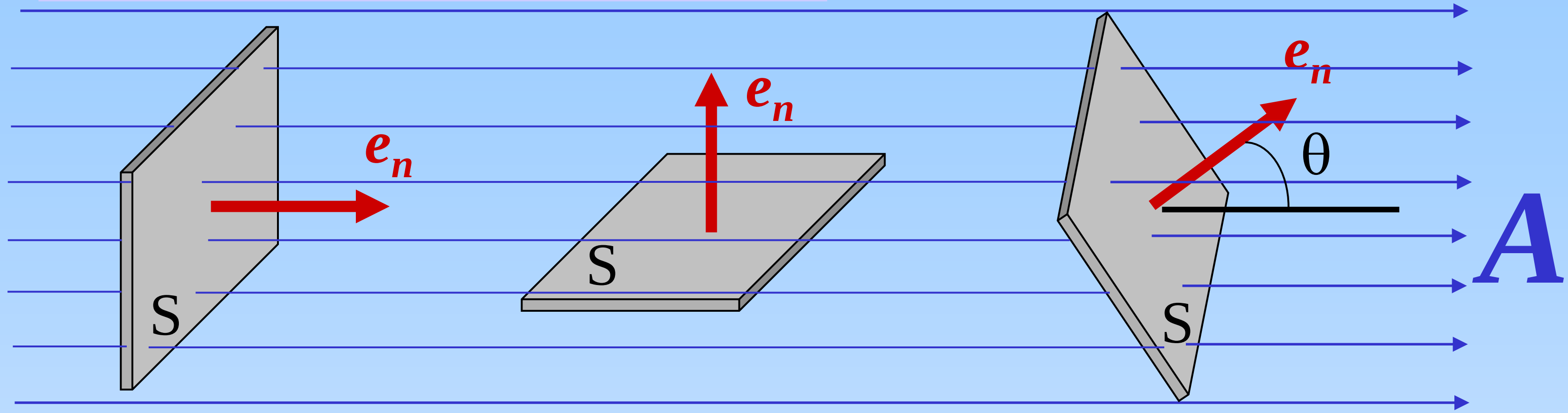
dV = du.dv.dw = (r.sinθ.dφ).(rdθ).dr

0 < φ < 2π ; 0 < θ < π

0 < r < R

5. Flux Φ : Definition

1. Homogeneous vector field *A*



$\theta \equiv \angle(\boldsymbol{e}_n, \boldsymbol{A}) = 0$

Def.: $\Phi = c.A.S$

Choice: $c \equiv 1$

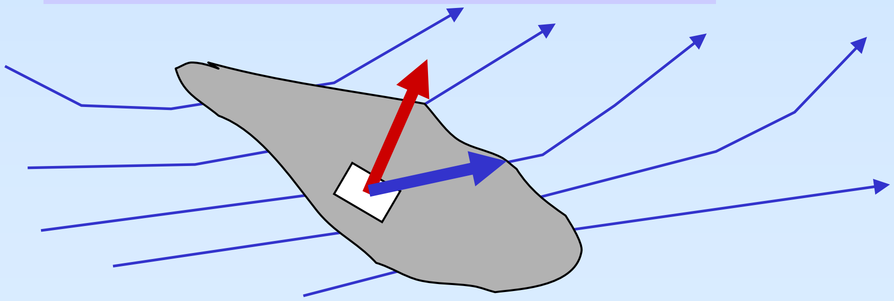
$\theta = 90^o$

$\Phi = 0$

$\Phi = A.S.\cos\theta$

$\Phi = (\boldsymbol{A}.\boldsymbol{e}_n)\, S$

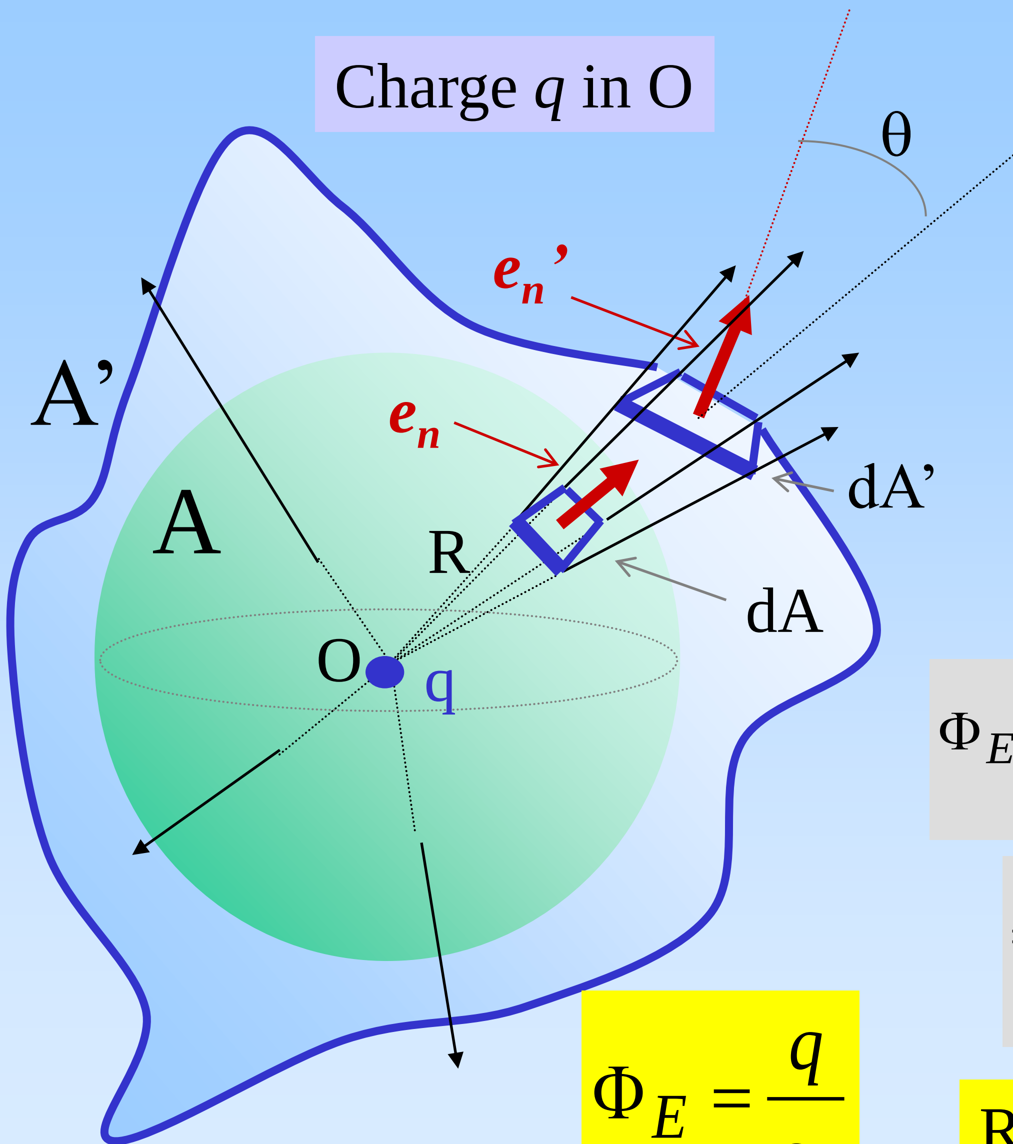
2. General vector field *A*



For small surface elements dS:
A and $\boldsymbol{e}_n$ are constant

$$\Phi = \iint_S \boldsymbol{A} \bullet \boldsymbol{e}_n dS = \iint_S \boldsymbol{A} \bullet \boldsymbol{dS}$$

6. Gauss' Law (1): derivation



Flux Φ_E through sphere A:

$$\Phi_E = \oiint_A \boldsymbol{E} \bullet \boldsymbol{dA} = \oiint_A \frac{q}{4\pi\varepsilon_0 r^2} \boldsymbol{e_r} \bullet \boldsymbol{e_n} dA$$

$$= \frac{q}{4\pi\varepsilon_0 R^2} .1.4\pi R^2 = \frac{q}{\varepsilon_0}$$

Flux Φ_E' through surface A':

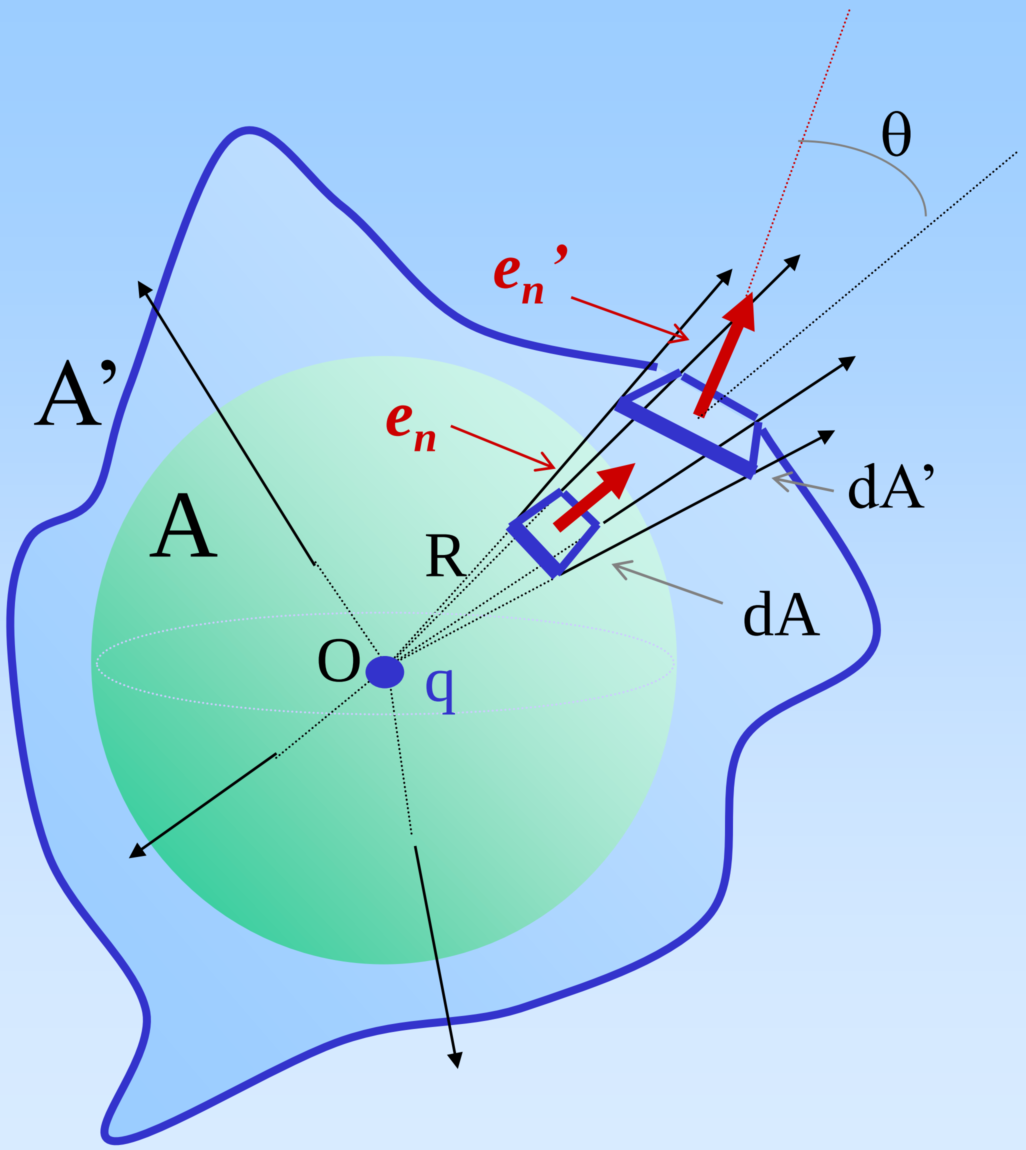
$$\Phi_E' = \oiint_{A'} \boldsymbol{E}' \bullet \boldsymbol{dA} = \oiint_{A'} \frac{\boldsymbol{q}}{4\pi\varepsilon_0 r'^2} \boldsymbol{e_{r'}} \bullet \boldsymbol{e_n}' dA'$$

$$= \oiint_{A'} \frac{q}{4\pi\varepsilon_0} \frac{\cos\theta . dA'}{r'^2} = \oiint_{A'} \frac{q}{4\pi\varepsilon_0} \frac{dA}{R^2} = \frac{q}{\varepsilon_0}$$

$$\Phi_E = \frac{q}{\varepsilon_0}$$

Result is independent of the shape of the surface

6. Gauss' Law (2): consequences



Flux:

$$\Phi_E = \frac{q}{\varepsilon_0}$$

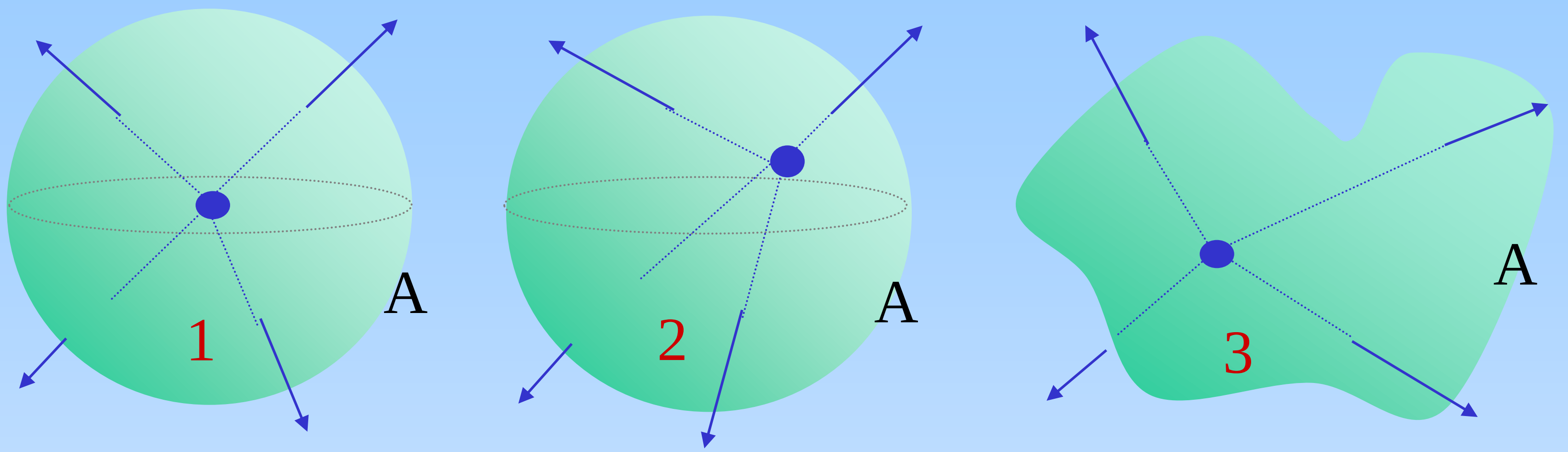
Result is independent of the shape of the surface

Consequences:

- q needs not to be in O
- charge outside: no net flux
- more charges in A:

$$\Phi_E = \sum_i \Phi_{E,i}$$

7. Gauss-boxes (1): symmetry

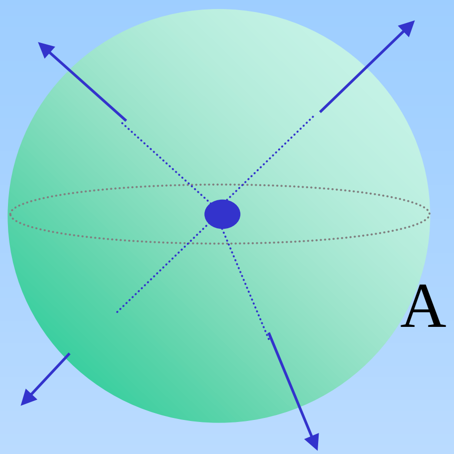


1: symmetry present: $E = E_{\perp} = \dfrac{q}{4\pi\varepsilon_0 R^2}$ everywhere

2+3: no symmetry: **averaged** $E_{\perp}$ over surface can be calculated only:

$$< E_{\perp} > = \frac{1}{A} \oint\!\!\oint_A \frac{q}{4\pi\varepsilon_0 r^2} \boldsymbol{e_r} \bullet \boldsymbol{e_n} dA$$

7. Gauss boxes (2) : solid sphere



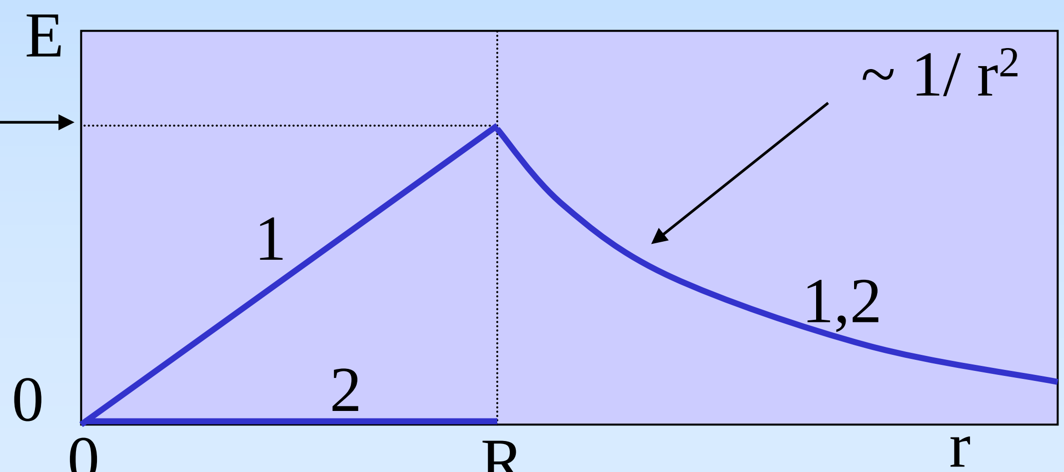
$$\Phi_E = \oint\!\!\oint_S \boldsymbol{E} \bullet \boldsymbol{dS} = \frac{q_{enclosed}}{\varepsilon_0}$$

$$E = E_\perp = \frac{q}{4\pi\varepsilon_0 R^2}$$

Solid sphere: radius R, charge q

1. Homogeneously distributed over volume (non-conducting)
2. Conducting: homogeneously distributed over surface

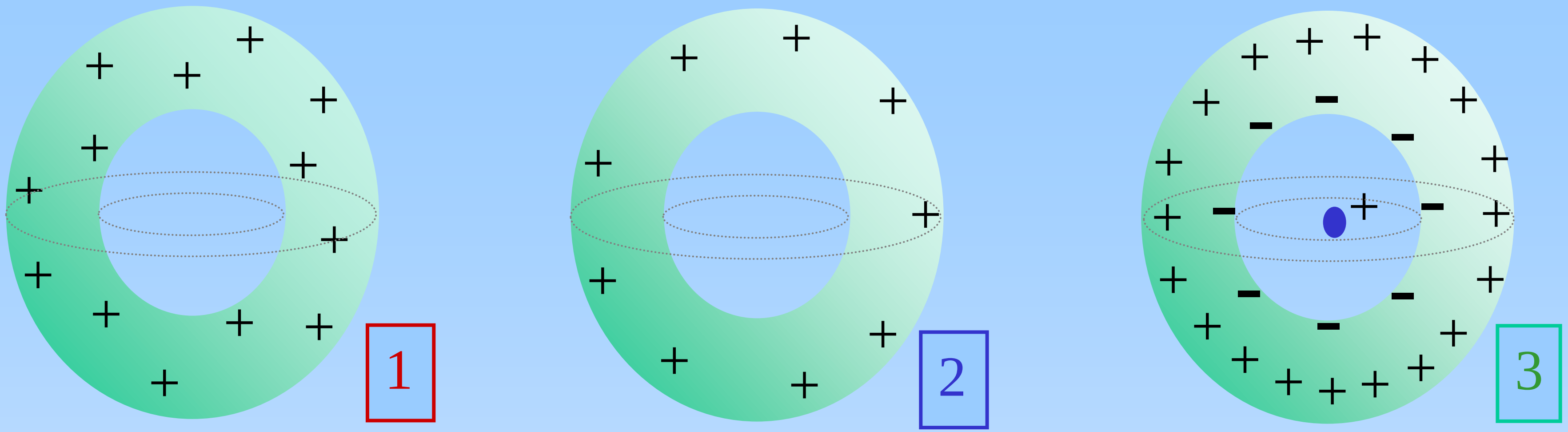
$$\frac{q}{4\pi\varepsilon_0 R^2}$$



Linear slope of 1. to be calculated using enclosed charge = $q \cdot (r/R)^3$ at radius r, and divide by surface $4\pi r^2$.

FdM

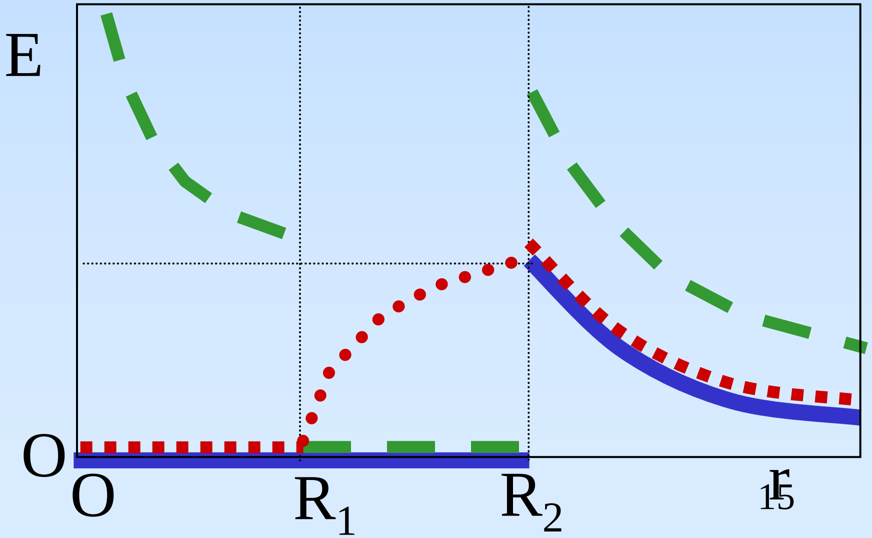
7. Gauss boxes (3) : hollow spheres



Radii: $R_1 < R_2$

Inside a conductor the field MUST be =0 (otherwise: rearrangement of charges)

1. Non-conducting;
   homog. charge Q
2. Conducting; charge Q
3. Idem, with extra q in O



7. Gauss-boxes (4): at plates

$$\oint\!\!\oint_A \boldsymbol{E} \bullet \boldsymbol{dA} = q / \varepsilon_0$$

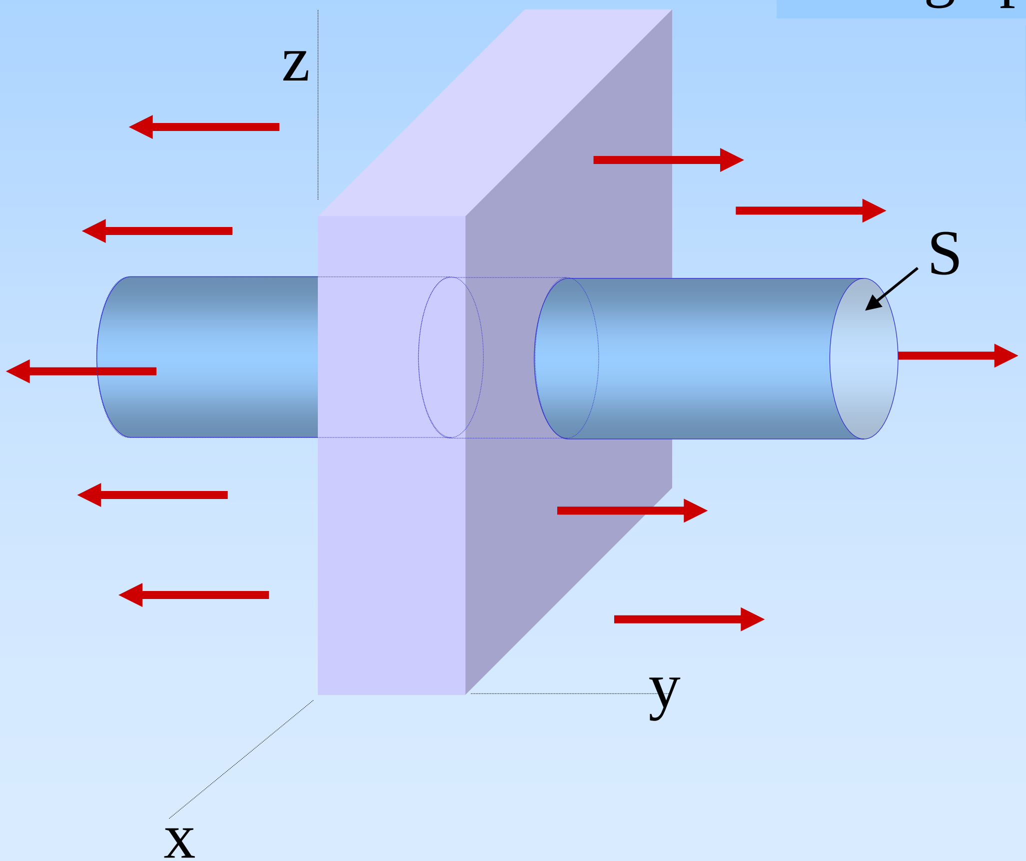
Convenient choices for surface A:

$\boldsymbol{E} = 0$ or $\boldsymbol{E} \,//\, \boldsymbol{dA}$ or $\boldsymbol{E} \perp \boldsymbol{dA}$

∞ large plate, homog. charge σ [C/m^2]

Symmetry: $\boldsymbol{E} \,//\, +\boldsymbol{e}_y$ resp. $-\boldsymbol{e}_y$

Take cylinder for Gauss box



Non-zero contributions to $\iint$ from top and bottom only

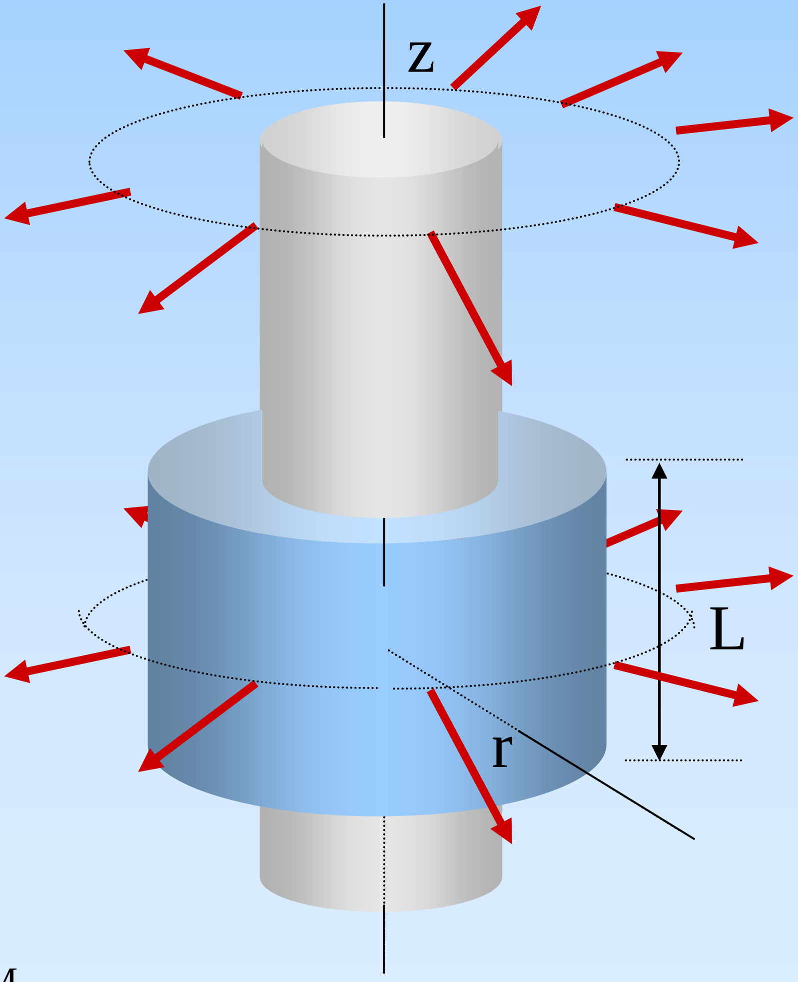
Enclosed charge: σ S

Gauss: $2.\,\boldsymbol{E}\,S = \sigma\,S / \varepsilon_0$

$$E = \tfrac{1}{2}\,\sigma / \varepsilon_0$$

7. Gauss-boxes (5): around rods

$$\oiint_A \boldsymbol{E} \bullet \boldsymbol{dA} = q / \varepsilon_0$$



Convenient choices for surface A:

$\boldsymbol{E} = 0$ or $\boldsymbol{E} // \boldsymbol{dA}$ or $\boldsymbol{E} \perp \boldsymbol{dA}$

∞ long bar, homog. charge λ [C/m]

Symmetry: ***E*** radial direction

Take cylinder for Gauss box

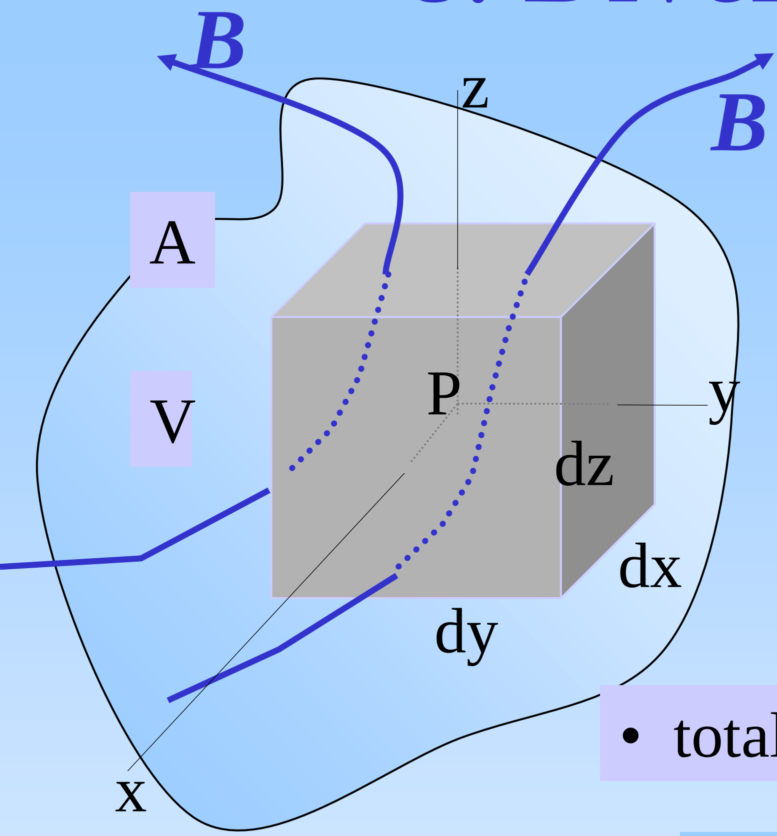
Non-zero contributions to ∬ from side wall only

Enclosed charge: λ L

Gauss: $\boldsymbol{E}.\ 2\pi r.L = \lambda\ L / \varepsilon_0$

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}$$

8. Divergence Theorem



Outgoing flux:

- Right side:

$$dB_{y,R} = \left[B_{P,y} + \frac{\partial B_y}{\partial y}\frac{dy}{2} \right] dx.dz$$

- Left side:

$$dB_{y,L} = -\left[B_{P,y} - \frac{\partial B_y}{\partial y}\frac{dy}{2} \right] dx.dz$$

+

- total in y-direction: $\frac{\partial B_y}{\partial y} dxdydz = \frac{\partial B_y}{\partial y} dV$

Total through the block :

$$d\Phi_B = \left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right] dV \equiv div\, \mathbf{B}.dV$$

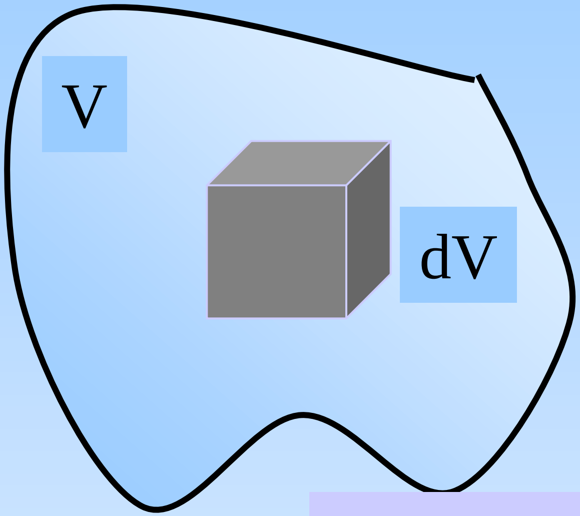
Integrate over volume V :

$$\Phi_B = \oiint_A \mathbf{B} \bullet \mathbf{dA} = \iiint_{\mathbf{V}} div\, \mathbf{B}.\mathbf{d}V$$

9. "Gauss" in differential form

Assume in V:
ρ(x,y,z) [C/m³]

$$d\Phi_E = \frac{dQ_{encl.}}{\varepsilon_0} = \frac{\rho . dV}{\varepsilon_0}$$



$$d\Phi_E = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] dV \equiv div\,\mathbf{E}.dV$$

"Gauss" in differential form:

$$div\,\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

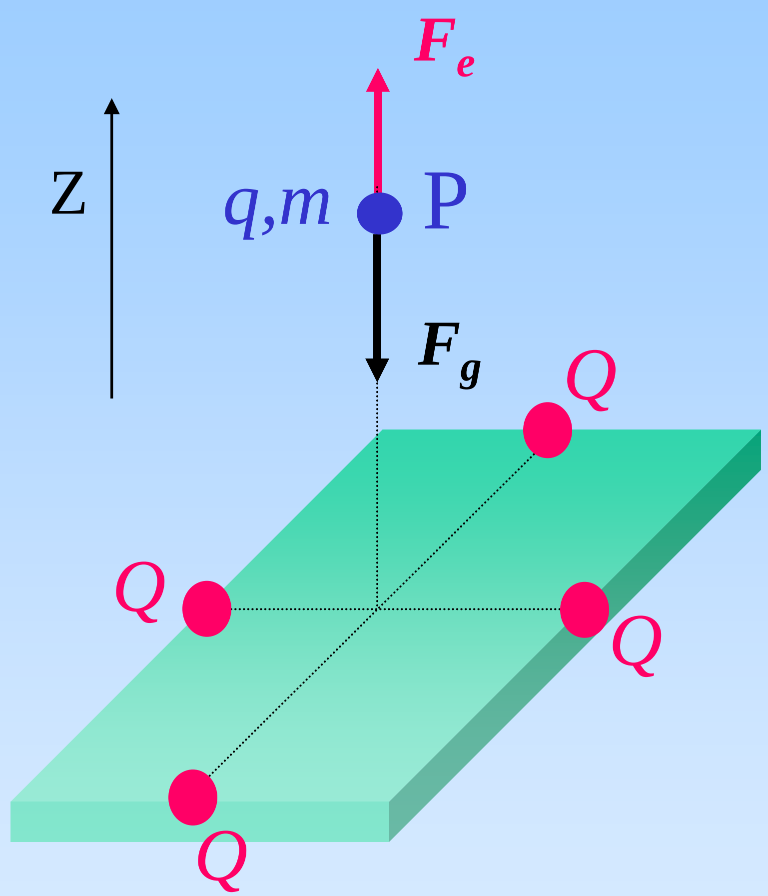
- Application: ρ (x,y,z) to be calculated from ***E*** (x,y,z)

- Divergence = "micro"-flux per unit of volume (m³) :

$$div\,\mathbf{E} = \frac{d\Phi_E}{dV}$$

$$\Phi_E = \oint\!\!\oint_A \mathbf{E} \bullet \mathbf{dA} = \iiint_{\mathbf{V}} div\,\mathbf{E}\,.\mathbf{d}V = \iiint_V \frac{\rho}{\varepsilon_0} dV = \frac{Q_{encl.}}{\varepsilon_0}$$

10. Electrostatic Crane (1)



Mass m with charge q

<u>Question:</u>

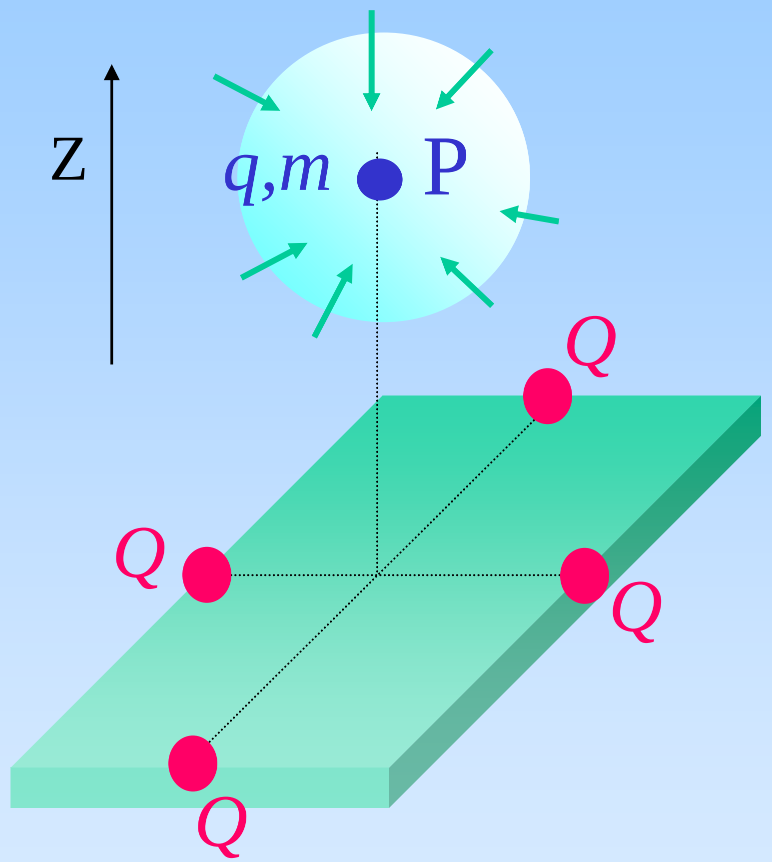
1. Can m be hold in equilibrium?
2. Can z_P be changed by changing q (or the Q's)?

<u>Approach:</u> Displace m to P' from equilibrium by (dx,dy,dz); Calculate direction of Coulomb-force $\boldsymbol{dF_e}$; From everywhere, $\boldsymbol{dF_e}$ should point to P

<u>Result:</u> for at least several (if not all ?) displacements:
direction of $\boldsymbol{dF_e}$ is outward from P.

<u>Conclusion: no stable equilibrium possible</u>

10. Electrostatic Crane (2)



From Coulomb:
No stable equilibrium possible.

Question: Does “Gauss” provide a “simple” proof?

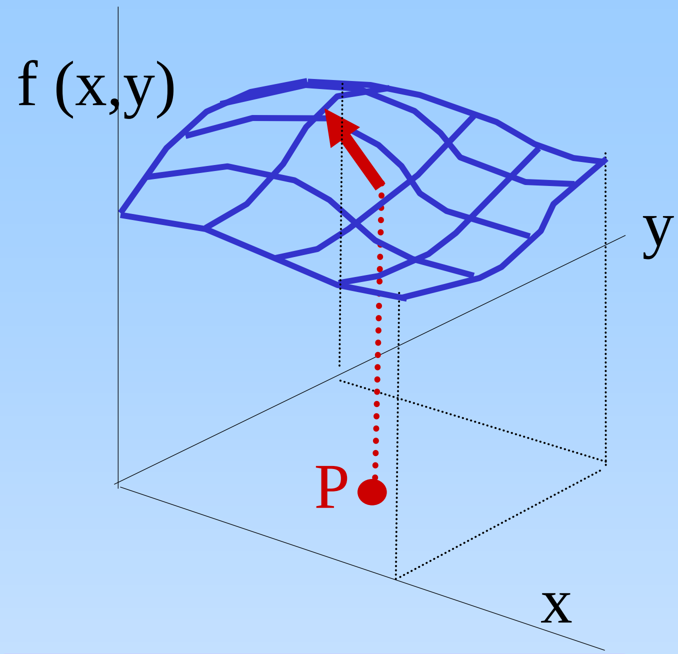
Approach: Gauss sphere around P

For stable equilibrium: all forces on P should point inward sphere

From “Gauss”: this is possible only when at P negative charge present !!

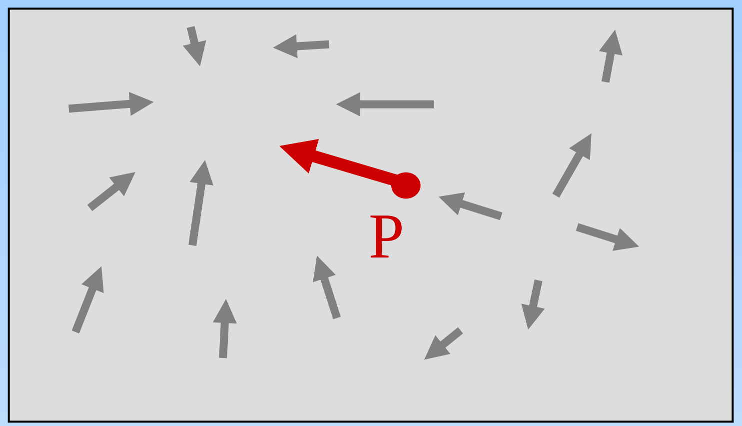
Conclusion: no negative charge at P $\Rightarrow$ no stable equilibrium

11. Gradient in 2 dimensions



Scalar function: f (x,y)

Top view: Steepest slopes:



Arrows represent vectorial function: grad f

$$grad\ \ f = \frac{\partial f}{\partial x}\boldsymbol{e_x} + \frac{\partial f}{\partial y}\boldsymbol{e}_y + \frac{\partial f}{\partial z}\boldsymbol{e}_z$$

$$\nabla\ f$$

12. Electric Potential (1): Definition

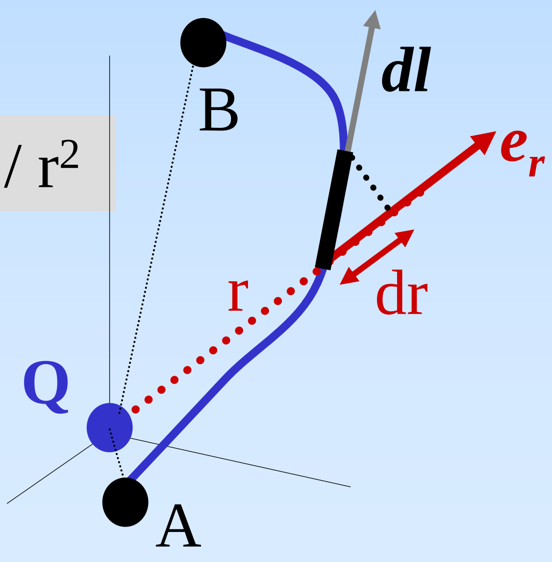
Field force $\boldsymbol{F_E}$

Work needed for transport of q, from A to B, “opposing field force”

$$W_{A\to B} = -\int_A^B \boldsymbol{F_E} \bullet \boldsymbol{dl} \text{ , met } \boldsymbol{F_E} = q\boldsymbol{E}$$

Work per Coulomb = potential difference

$$\frac{W_{A\to B}}{q} = -\int_A^B \boldsymbol{E} \bullet \boldsymbol{dl} \equiv V_{A\to B} \equiv V_B - V_A$$



$\boldsymbol{E} \sim \boldsymbol{e_r} / r^2$

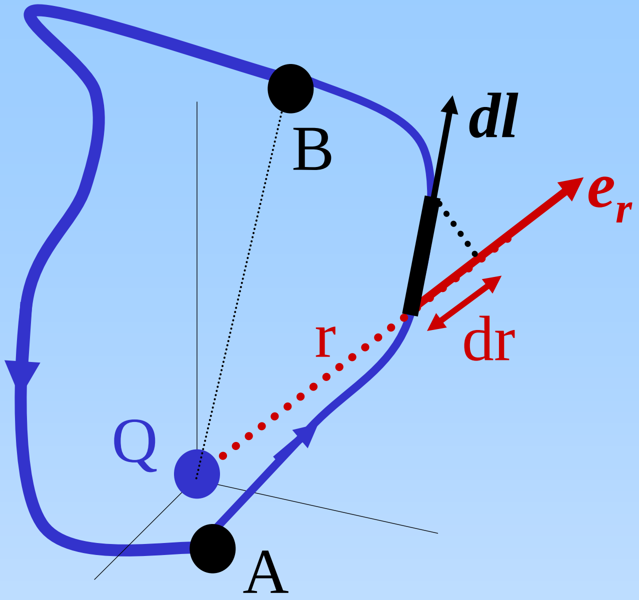
$$\boldsymbol{E} \bullet \boldsymbol{dl} = \frac{Q\, \boldsymbol{e_r} \bullet \boldsymbol{dl}}{4\pi\varepsilon_0 r^2} = \frac{Q\, dr}{4\pi\varepsilon_0 r^2}$$

$$V_B - V_A = -\int_A^B \boldsymbol{E} \bullet \boldsymbol{dl} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

Work is independent of path choice

Conservative field

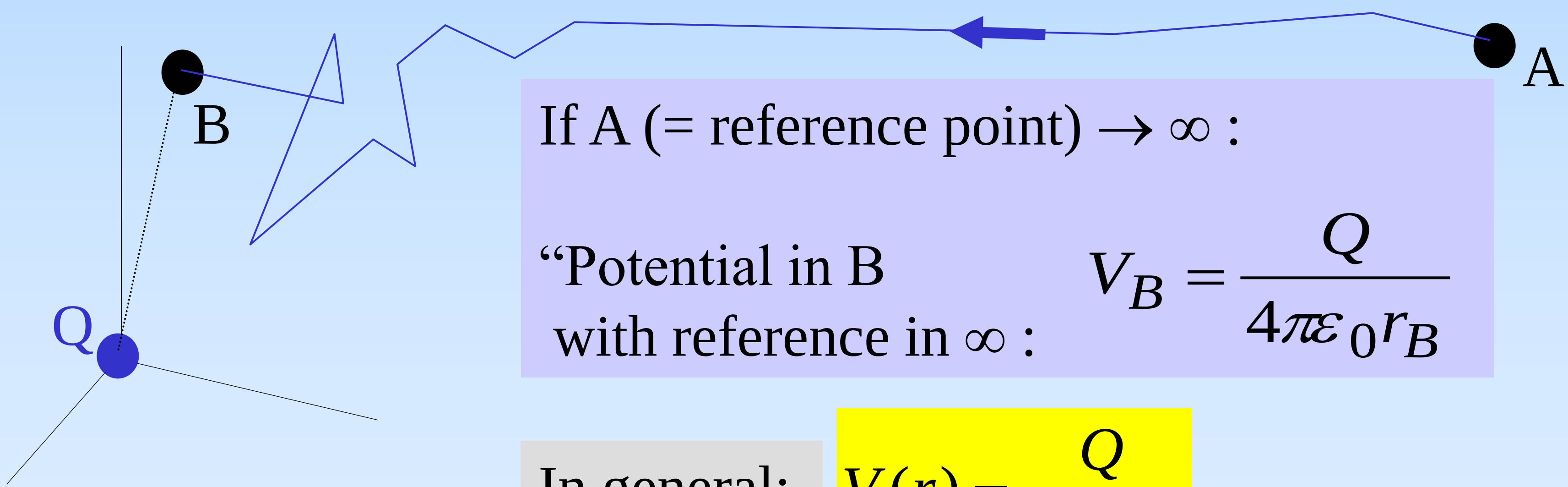
12. Electric Potential (2): Reference



$$V_B - V_A = -\int_A^B \boldsymbol{E} \bullet \boldsymbol{dl} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$\oint \boldsymbol{E} \bullet \boldsymbol{dl} = 0 \quad \Rightarrow \quad \nabla \times \boldsymbol{E} = 0$$

Circulation-free field



If A (= reference point) $\rightarrow \infty$:

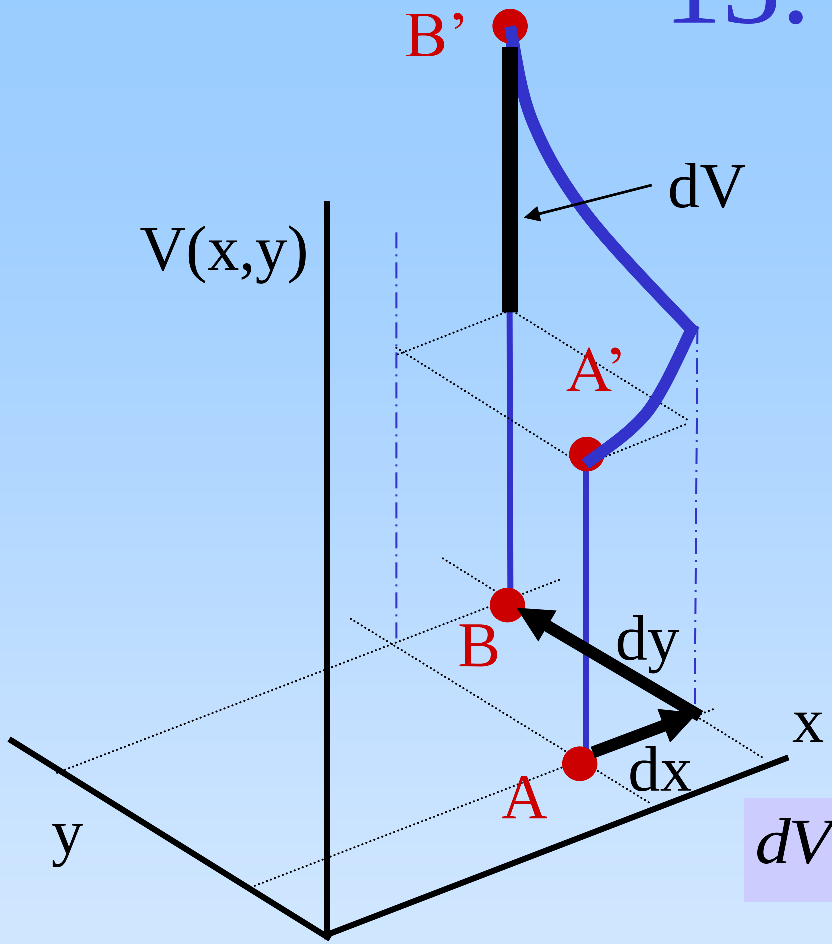
"Potential in B with reference in ∞ :

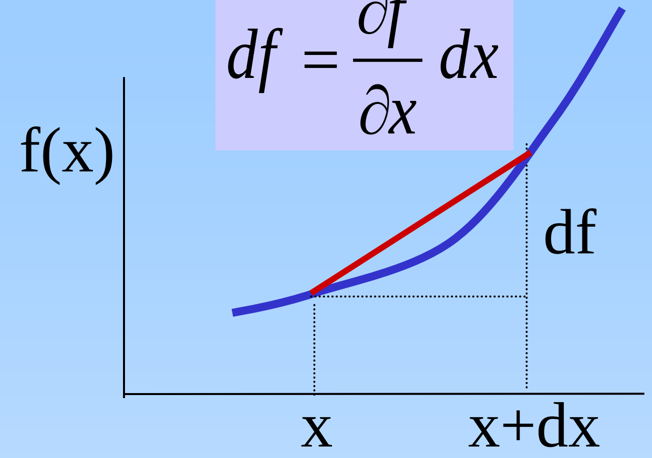
$$V_B = \frac{Q}{4\pi\varepsilon_0 r_B}$$

In general:

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

13. Field and Potential





$$df = \frac{\partial f}{\partial x} dx$$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \left(\frac{\partial V}{\partial z} dz \right)$$

$$dV = V_B - V_A = -E_x.dx - E_y.dy \left(-E_z.dz \right)$$

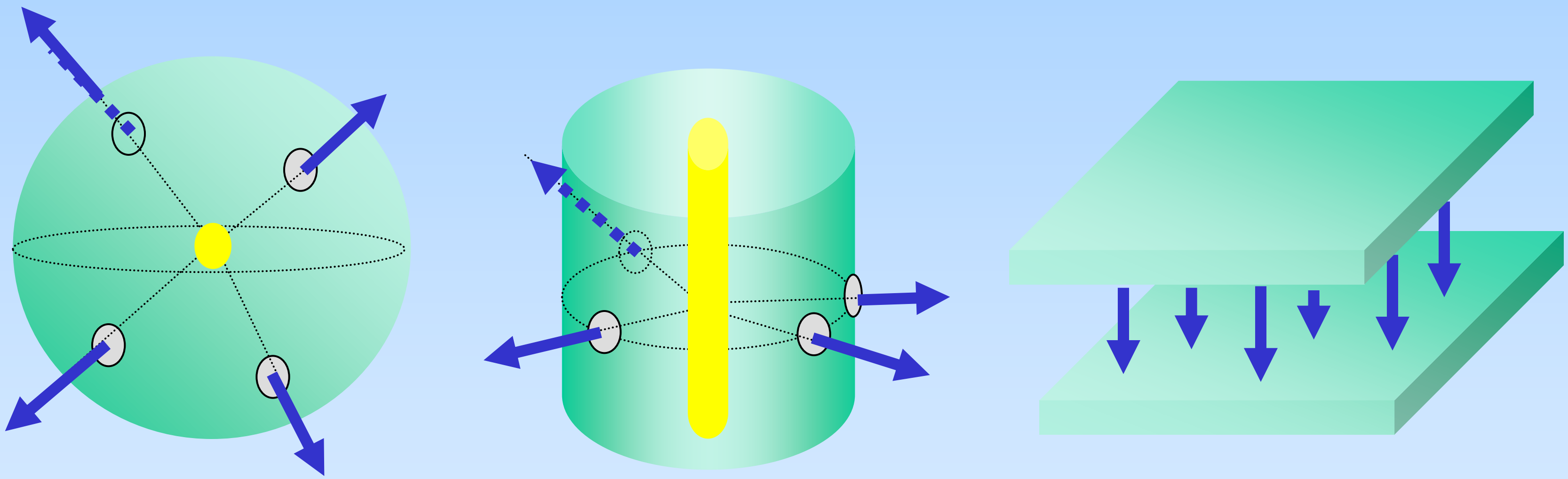
$$grad\ f = \frac{\partial f}{\partial x} \boldsymbol{e_x} + \frac{\partial f}{\partial y} \boldsymbol{e_y} + \frac{\partial f}{\partial z} \boldsymbol{e_z}$$

$$V_B - V_A = -\int_A^B \boldsymbol{E} \bullet \boldsymbol{dl}$$

$$\boldsymbol{E} = -grad\ V$$

14. Vector fields: rotation-free

$$\oint \boldsymbol{A} \bullet \boldsymbol{dl} = 0 \;\Leftrightarrow\; \nabla \times \boldsymbol{A} = 0$$



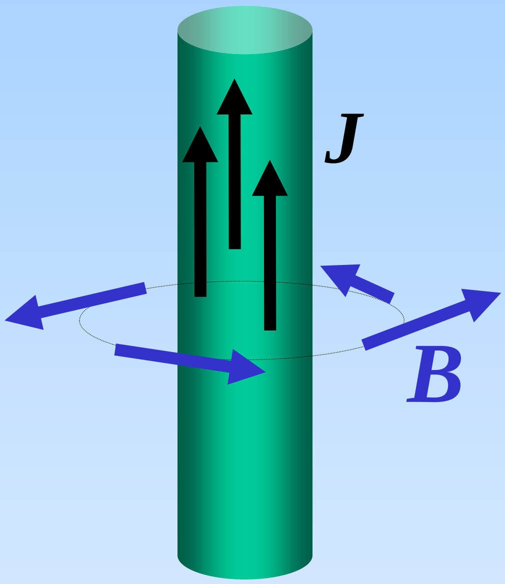
$A \sim r^{-2}$

$A \sim r^{-1}$

$A \sim r^{0} = c$

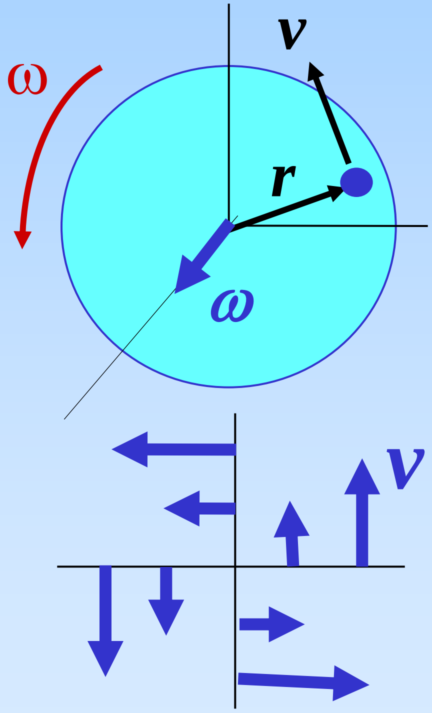
15.Vector fields: non-zero rotation

$$\oint \boldsymbol{A} \bullet \boldsymbol{dl} \neq 0 \iff \nabla \times \boldsymbol{A} \neq 0$$

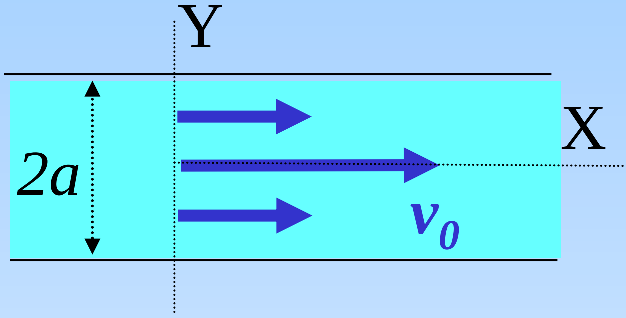


$$\oint \boldsymbol{B} \bullet \boldsymbol{dl} = \mu_0 I$$

$$rot\ \boldsymbol{B} = \mu_0 \boldsymbol{J}$$



$$rot\ \boldsymbol{v} = 2\omega$$

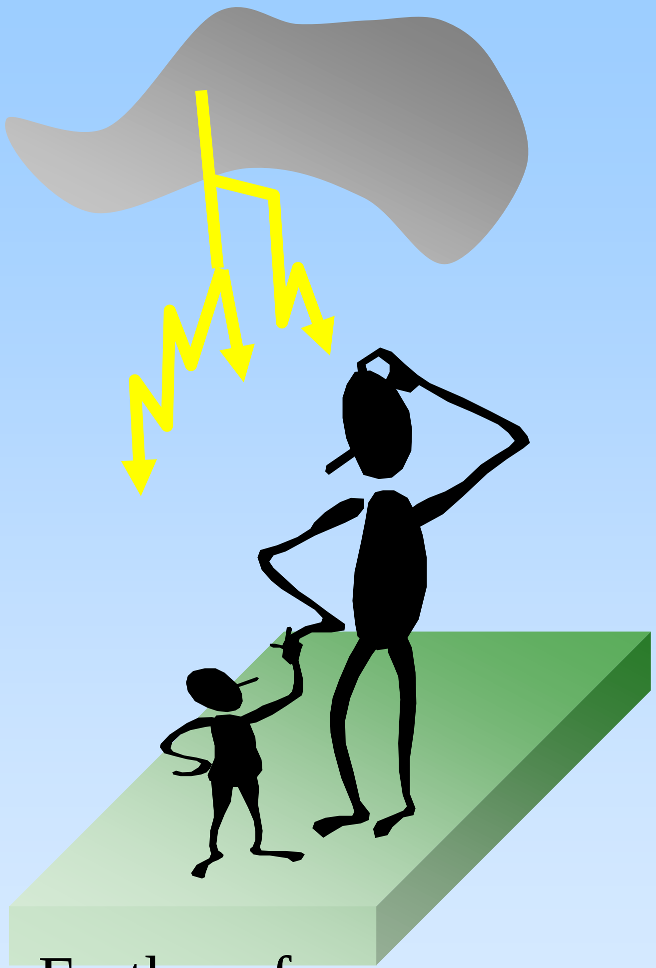


River flow

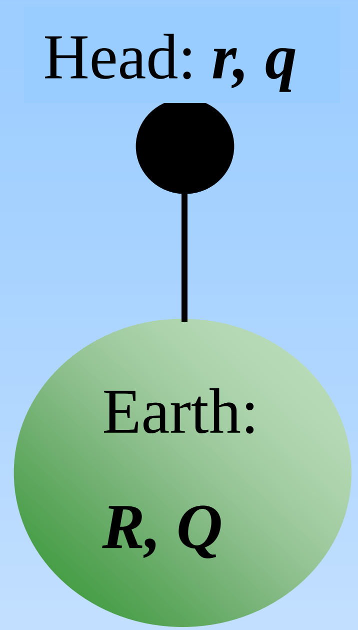
If linear profile:

$$rot\ v = \pm \frac{v_0}{a} \boldsymbol{e_z}$$

16. Lightning: greatest risk ?



Earth surface
Equipotential plane



Sphere:

$$E(r) \sim qr^{-2}\,; V(r) \sim qr^{-1}$$

Equipotential $\Rightarrow$

$$\frac{Q}{R} = \frac{q}{r} \Rightarrow \frac{q}{Q} = \frac{r}{R}$$

$$\frac{E_{head}}{E_{earth}} = \frac{q/r^2}{Q/R^2} = \frac{q.R^2}{Q.r^2} = \frac{R}{r} >> 1$$

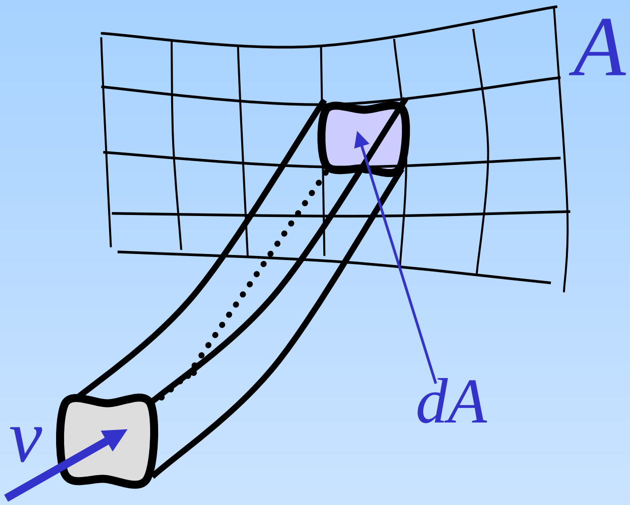
$\Rightarrow$ Smallest head, largest ***E***-field, thus greatest risk.

$\Rightarrow$ Lightning conductor has to have a sharp point (small radius).

17. Particle transport and Flux

Particles: density n per m^3 ; velocity v m/s (position dependent)

Flow tube: contains all particles going through dA



Flux Φ: nr. of particles passing per sec.

All particles within v meter from A will pass within 1 sec. from now:

$$d\Phi = nv.dA$$

$$\text{Particle flux [part./sec]} : \Phi_n = \iint_A n\,\vec{v}\,d\vec{A}$$

$n.v$ = flux density [part. m^{-2}s^{-1}]

Suppose all particles carry charge q :

Charge flux = current:

$$\Phi_q = \iint_A n\vec{\mathbf{v}}q.d\vec{\mathbf{A}} = \iint_A \vec{\boldsymbol{j}} \bullet \boldsymbol{d}\vec{\mathbf{A}} = I = q.\Phi_n$$

$\boldsymbol{j}$ = current density

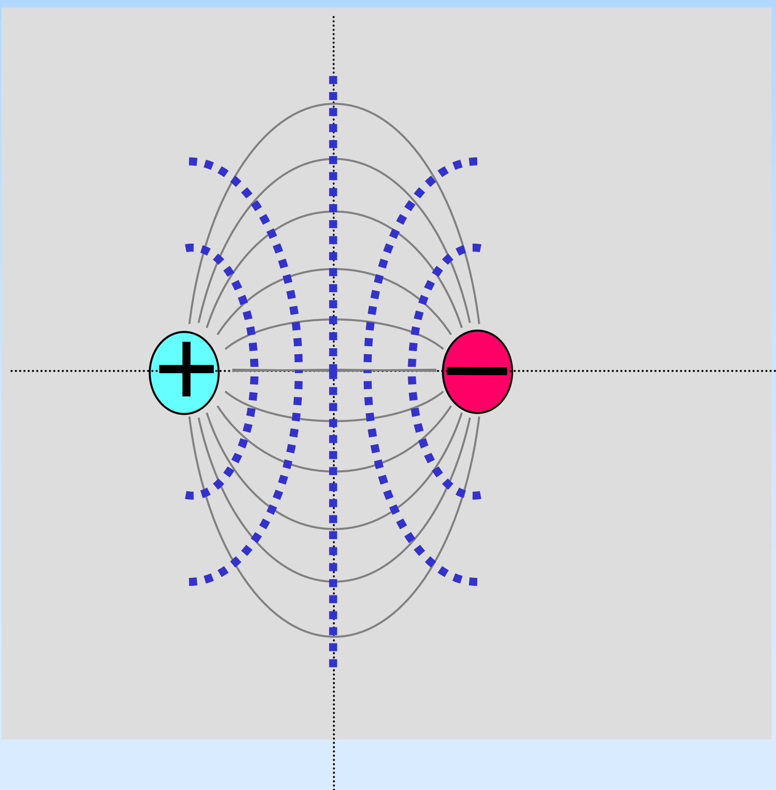
FdM

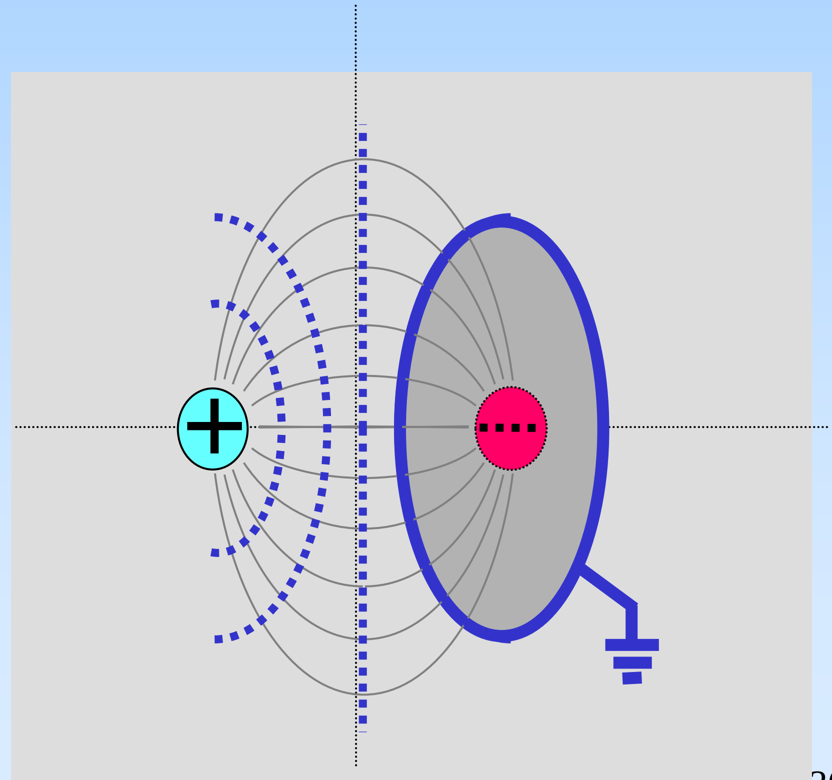
18. Point Charge opposing Conductor

Field lines (——) and equipotential surfaces (----) are mutually perpendicular, since if $\Delta V=0$, $\int \boldsymbol{E.dl} = 0$ and $\boldsymbol{E} \perp \boldsymbol{dl}$

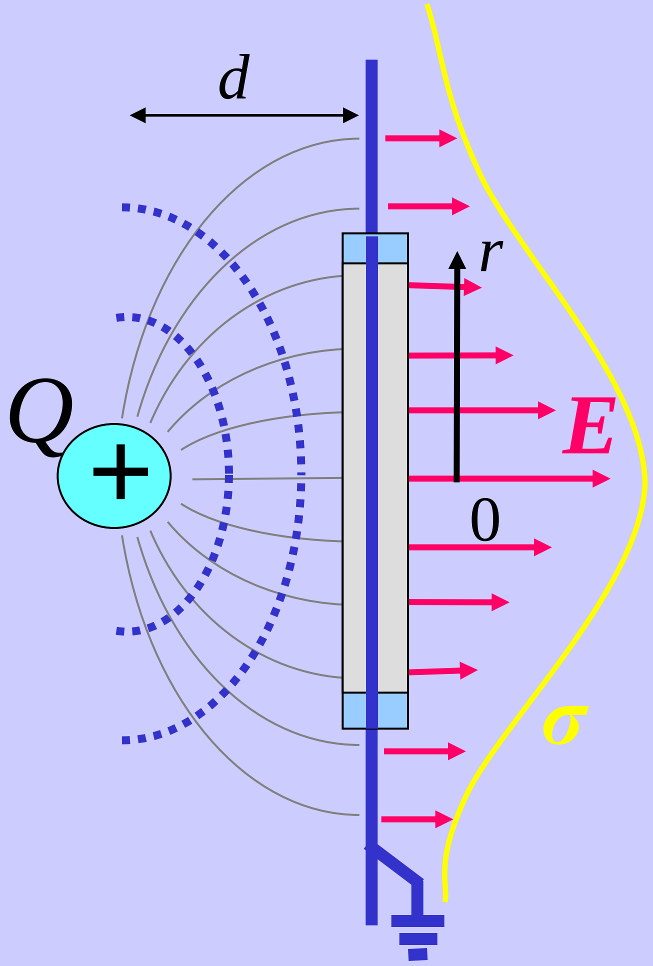
<u>Opposing charges:</u>
dipolar field

<u>Charge opposing conductor:</u>
image charge inside conductor





19. Point charge vs. flat conductor



Cross section

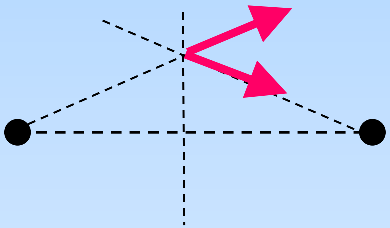
<u>Question:</u> determine $\sigma(r)$ [C/m^2]

Gauss box: ring-shaped box, radii: r and $r+dr$

$$\text{Gauss}: \; E.2\pi r.dr = \frac{1}{\varepsilon_0}\sigma(r).2\pi r.dr$$

$$\Longrightarrow \quad \sigma(r) = \varepsilon_0 \, E(r)$$

Apply image charge and Coulomb:



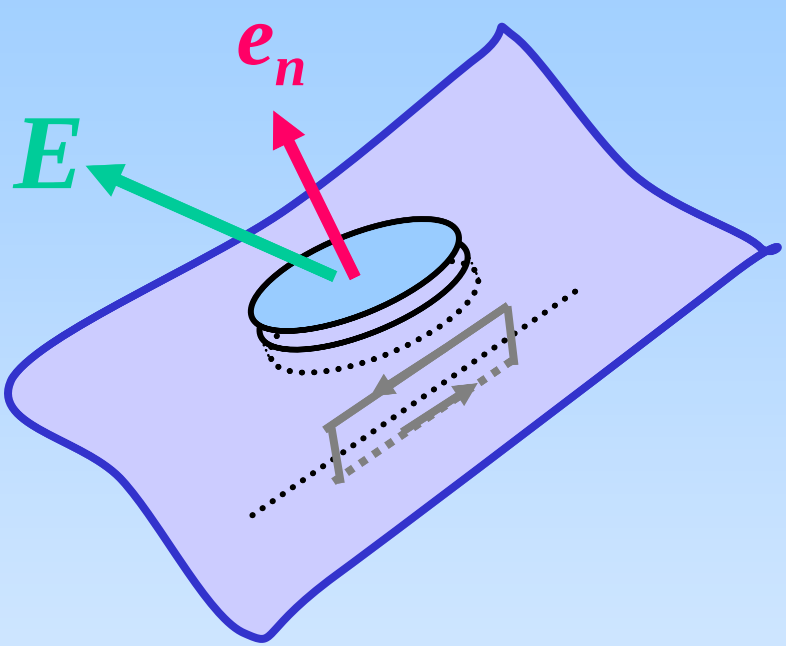
$$E(r) = \frac{Q}{4\pi\varepsilon_0[d^2+r^2]^2}\frac{2d}{\sqrt{d^2+r^2}}$$

$$\Rightarrow \sigma(r) = \varepsilon_0 E(r) = \frac{Qd}{2\pi[d^2+r^2]^{3/2}}$$

FdM

20. Electric field at Interface

Surface charge: σ [C/m^2]



Slope of V along normal unit vector on surface :

Thin Gauss box:
Top and bottom lid: $E_{\perp\uparrow}$ and $E_{\perp\downarrow}$
(normal components)

$E_{\perp\uparrow} - E_{\perp\downarrow} = \sigma/\varepsilon_0$

Rectangular loop:

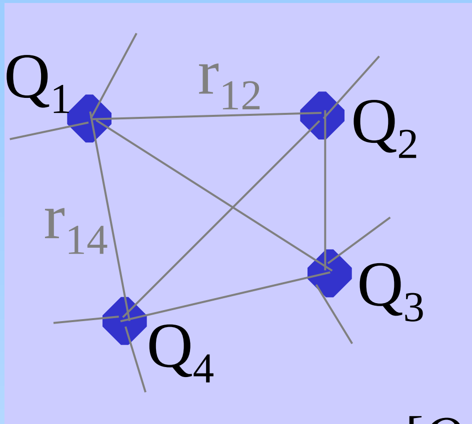
$\oint \boldsymbol{E} \bullet \boldsymbol{dl} = 0 \Rightarrow \qquad E_{//\uparrow} = E_{//\downarrow}$

$$\boldsymbol{E}_{\uparrow} - \boldsymbol{E}_{\downarrow} = \frac{\sigma}{\varepsilon_0}\boldsymbol{e_n}$$

$$\boldsymbol{E} = -\nabla V \Rightarrow \quad \nabla V \bullet \boldsymbol{e_n} = \frac{\partial V}{\partial n} \quad \Rightarrow$$

$$\frac{\partial V_{\uparrow}}{\partial n} - \frac{\partial V_{\downarrow}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

21. Energy of/in Charge Distribution



Energy for creation = energy for destruction

Potential in field of charge Q_i : $V_i = \frac{Q_i}{4\pi\varepsilon_0 r}$

Total Energy released upon destruction : E_{tot}

$r_{ij}=r_{ji}$

$$[Q_1 \to \infty] \quad 2E_{tot} = \frac{Q_1}{4\pi\varepsilon_0}\left[0 + \frac{Q_2}{r_{12}} + \frac{Q_3}{r_{13}} + \frac{Q_4}{r_{14}} + \ldots\ldots + \frac{Q_N}{r_{1N}}\right] +$$

$$[Q_2 \to \infty] \quad + \frac{Q_2}{4\pi\varepsilon_0}\left[\frac{Q_1}{r_{21}} + 0 + \frac{Q_3}{r_{23}} + \frac{Q_4}{r_{24}} + \ldots\ldots + \frac{Q_N}{r_{2N}}\right] +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$[Q_N \to \infty] \quad + \frac{Q_N}{4\pi\varepsilon_0}\left[\frac{Q_1}{r_{N1}} + \frac{Q_2}{r_{N2}} + \frac{Q_3}{r_{N3}} + \frac{Q_4}{r_{N4}} + \ldots\ldots + 0\right]$$

$$2E_{tot} = Q_1V_1 + Q_2V_2 + \ldots\ldots + Q_NV_N$$

$$E_{tot} = \frac{1}{2}\sum_i Q_iV_i$$

with $V_i = \sum_{j\neq i}\frac{Q_j}{4\pi\varepsilon_0 r_{ji}}$

22. Energy to Charge a Capacitor

Question: Energy to charge capacitor $\boldsymbol{C}$ up to end charge $\boldsymbol{Q_E}$

Suppose: "Now" already charge $\boldsymbol{Q}$ present $\Rightarrow$ potential $\boldsymbol{V = Q / C}$

Adding charge $\boldsymbol{q}$: does this cost extra energy $\boldsymbol{q.V}$?

No, because during addition $\boldsymbol{V}$ will change! $\boldsymbol{V = f(Q)}$

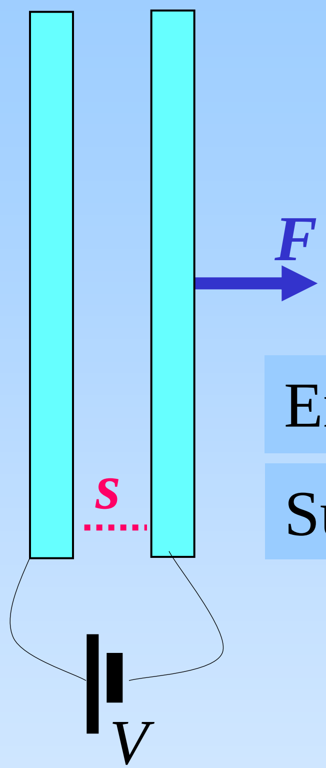
Approach: add $\boldsymbol{dQ}$; then $\boldsymbol{V}$ remains constant ($\boldsymbol{dV \ll V}$)

This will cost energy: $\boldsymbol{dE = V.dQ = (Q / C) \,.\, dQ}$

In total, charge from $\boldsymbol{0}$ to $\boldsymbol{Q_E}$:

$$E_{tot} = \int_0^{Q_E} \frac{Q}{C} dQ = \frac{1}{2} \frac{Q_E{}^2}{C} = \frac{1}{2} Q_E . V_E \qquad V_E = \frac{Q_E}{C}$$

23. Change Spacing in a flat Capacitor



Suppose: capacitor connected to battery, potential ***V***

Question: Energy and force to change spacing ***s*** to ***s+ds***

Important variables: ***V*** (=const.) ; ***C, Q, E*** all ***f (s)***

Energy balance: net supplied energy = growth of field energy

Supplied: mechanical and electrical (from battery)

$$F.ds + V.dQ = d(\tfrac{1}{2}\varepsilon_0 E^2.As) \quad ; \quad As = \text{volume}$$

$$V.dQ = V^2 dC = V^2 d(\varepsilon_0 A/s) = -\varepsilon_0 V^2 A ds/s^2$$

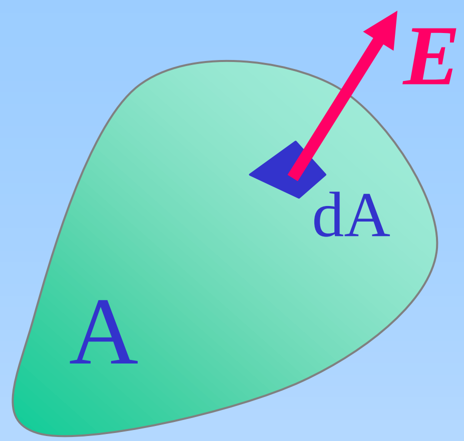
$$d(\tfrac{1}{2}\varepsilon_0 E^2.As) = d(\tfrac{1}{2}\varepsilon_0 \{V/s\}^2.As) = -\tfrac{1}{2}\varepsilon_0 V^2 A\frac{ds}{s^2}$$

$$F = +\frac{1}{2}\varepsilon_0 V^2 A/s^2$$

~ *A* ; ~ *s* $^{-2}$ **; ~ *V*** ^{2}

FdM

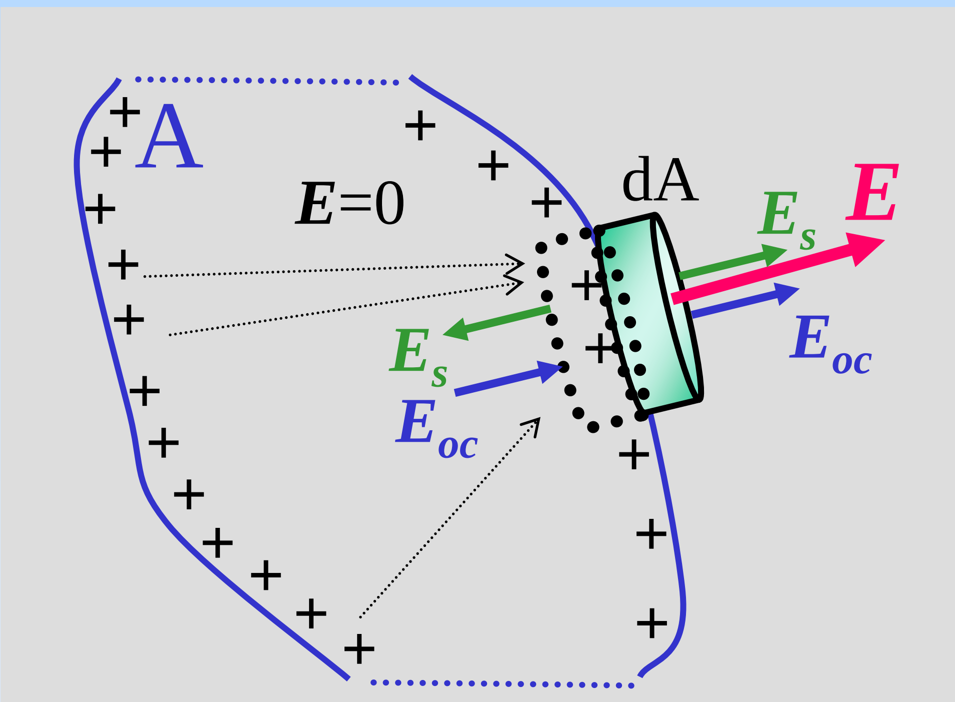
24. Conductor: Field at Boundary



Everywhere on A: $\boldsymbol{E} \perp \boldsymbol{dA}$

Suppose charge density σ [C/m^2] is f (position)

Inside conductor: field must be =0 !



Gauss: $\boldsymbol{E} \cdot \boldsymbol{dA} = \sigma\, dA / \varepsilon_0$

$E = \sigma / \varepsilon_0$

Self-field of box dA:

$E_s = \sigma / 2\varepsilon_0$

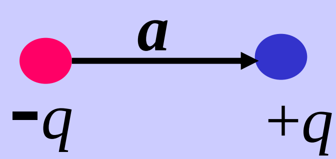
Field from other charges outside box dA:

$E_{oc} = \sigma / 2\varepsilon_0$

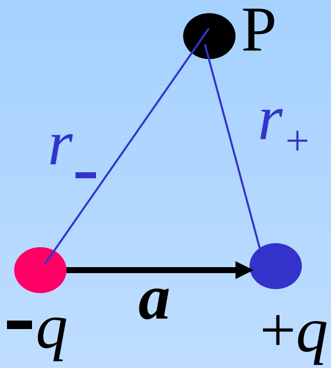
FdM

25. Electric Dipole (1): Far-field

Definition:

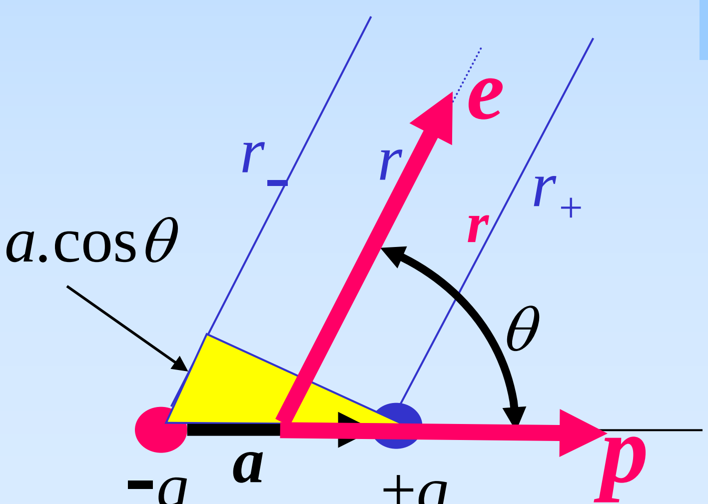


Dipole moment: $\boldsymbol{p} = q\,\boldsymbol{a}$



Potential field (with respect to ∞) :

$$V_P = \frac{+q}{4\pi\varepsilon_0 r_+} + \frac{-q}{4\pi\varepsilon_0 r_-} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$

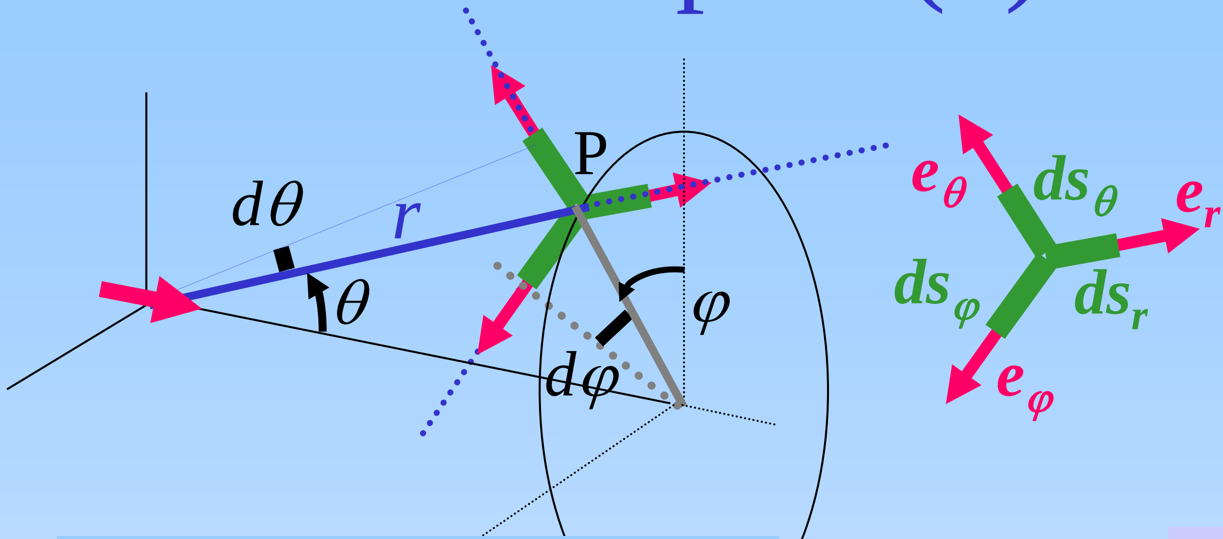


Far-field approximation: P → ∞

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{r_- - r_+}{r_+ r_-} \approx \frac{a\cos\theta}{r^2}$$

$$V_P = \frac{qa.\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p.\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\boldsymbol{p}\bullet\boldsymbol{e_r}}{4\pi\varepsilon_0 r^2}$$

25. Electric Dipole (2): components



$$V_P = \frac{p.\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\boldsymbol{E} = E_r\,\boldsymbol{e_r} + E_\theta\,\boldsymbol{e_\theta} + E_\varphi\,\boldsymbol{e_\varphi}$$

$$\boldsymbol{E} = -\,\mathrm{grad}\,V$$

$$ds_r = dr \quad\Rightarrow\quad E_r = -\frac{\partial V}{\partial s_r} = -\frac{\partial V}{\partial r}\frac{dr}{ds_r} = \frac{2p.\cos\theta}{4\pi\varepsilon_0 r^3}.1 = \frac{2p.\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$ds_\theta = r.d\theta \quad\Rightarrow\quad E_\theta = -\frac{\partial V}{\partial s_\theta} = -\frac{\partial V}{\partial \theta}\frac{d\theta}{ds_\theta} = \frac{p.\sin\theta}{4\pi\varepsilon_0 r^2}\frac{1}{r} = \frac{p.\sin\theta}{4\pi\varepsilon_0 r^3}$$

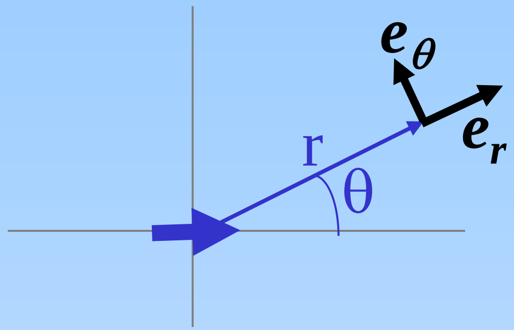
$$ds_\varphi = r.\sin\theta.d\varphi \quad\Rightarrow\quad E\varphi = -\frac{\partial V}{\partial s_\varphi} = -\frac{\partial V}{\partial \varphi}\frac{d\varphi}{ds_\varphi} = 0\,\frac{1}{r.\sin\theta} = 0$$

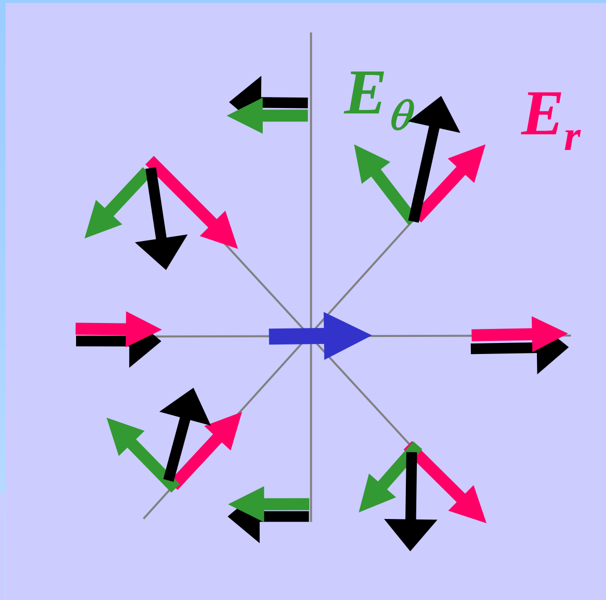
25. Electric Dipole (3): field lines

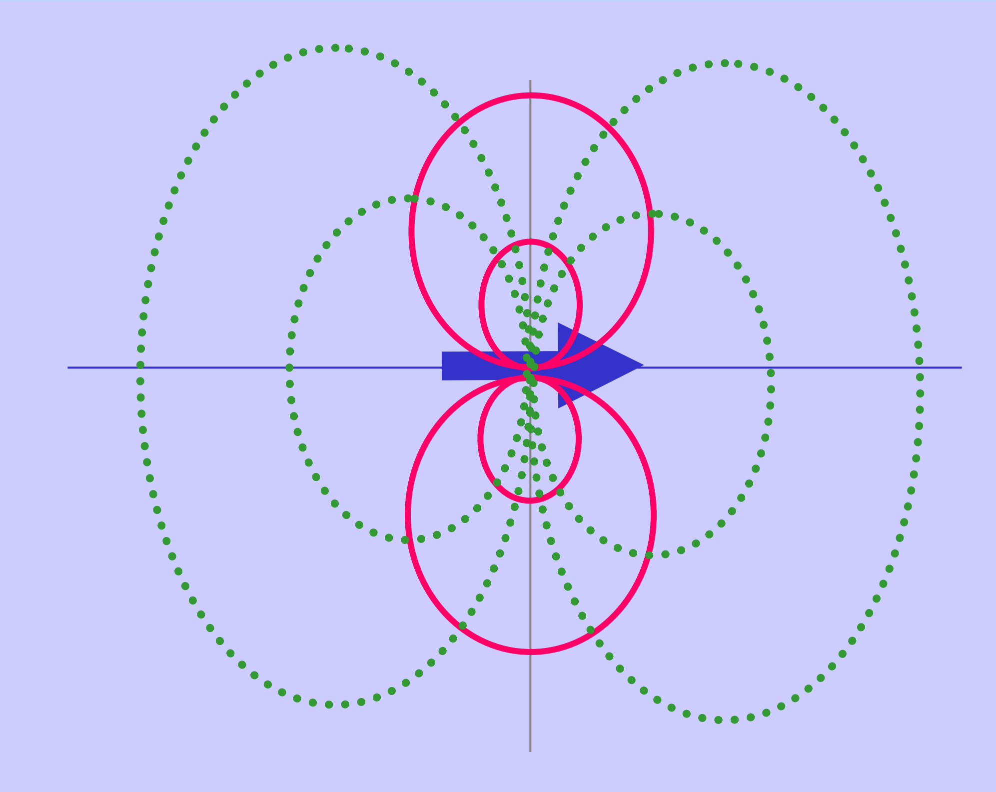
$$\boldsymbol{E} = E_r\,\boldsymbol{e_r} + E_\theta\,\boldsymbol{e_\theta}$$

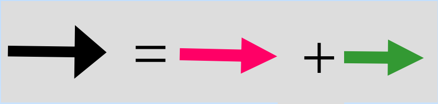
$$E_r = \frac{2p.\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = \frac{p.\sin\theta}{4\pi\varepsilon_0 r^3}$$







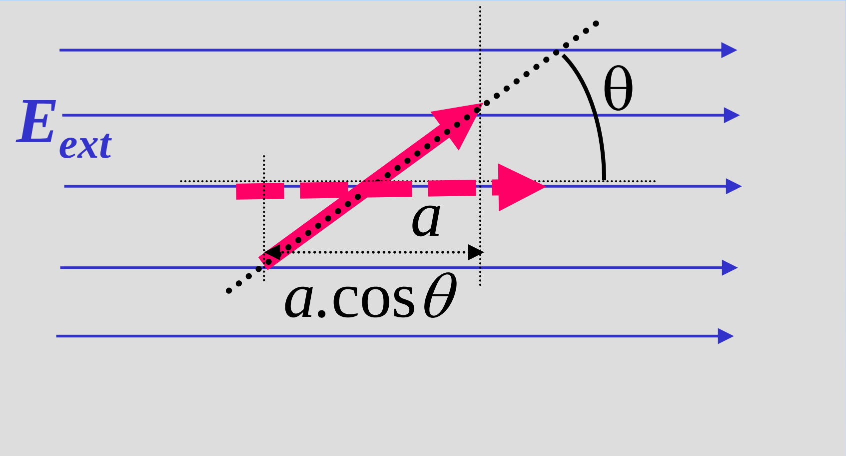


Field lines perpendicular to equipotential planes (where $\boldsymbol{E.dl} = 0$)

Electric Dipole (4): potential energy

Potential energy of a dipole in an homogeneous external field $\boldsymbol{E}_{ext}$

Question: Work needed to rotate dipole from θ = 0 to θ



Dipole: $\boldsymbol{p} = q\,\boldsymbol{a}$

Energetically most favourable orientation: $\theta = 0$
Minimum energy level E(0).

Work: E(θ) -E(0) = $-\int_0^\theta \boldsymbol{F} \bullet \boldsymbol{ds} = 2qE_{ext}\,\frac{1}{2}\,a.(1-\cos\theta) = pE_{ext}(1-\cos\theta)$

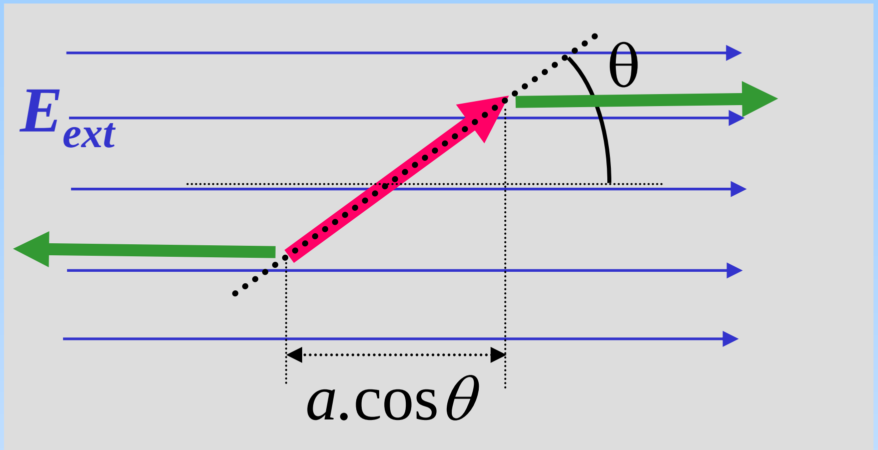
Set zero level: E(0) = - pE_{ext}

Potential energy: E(θ)= - pE_{ext}.cos θ

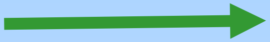
E(θ)= - $\boldsymbol{p} \bullet \boldsymbol{E}_{ext}$

25. Electric Dipole (5): torque

<u>Torque</u> of a dipole in an homogeneous external field $\boldsymbol{E}_{ext}$



Dipole: $\boldsymbol{p} = q\,\boldsymbol{a}$



Electrical force $\boldsymbol{F}_E$ on + and - charge

Torque at angle θ:

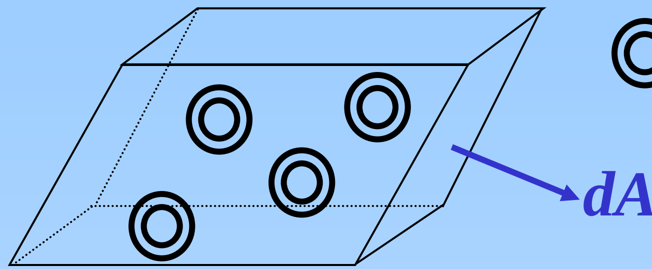
$$|\boldsymbol{\tau}| = 2\,F_E\,.\,\tfrac{1}{2}\,a.\,\sin\,\theta = p\,E_{ext}\,\sin\,\theta$$

Direction given by right-hand rule:

$$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}_{ext}$$

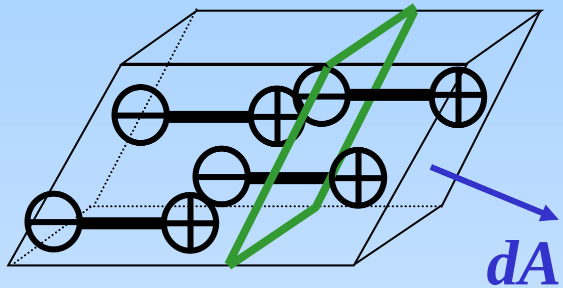
26. Polarization of a Dielectric

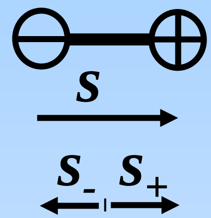
$\boldsymbol{E}_{ext} = 0$



◎ Unpolarized molecule
(n mol / m^3)

$\boldsymbol{E}_{ext} \neq 0$





Separation of charges

Dipole: $\boldsymbol{p} = q\,\boldsymbol{s}$

$\boldsymbol{s} = \boldsymbol{s}_+ - \boldsymbol{s}_-$

Charge transport through internal surface element ***dA*** :

$dQ = N_+\,Q - N_-(-Q) = (N_+ + N_-)Q = n.VQ$, with $V = vol$ (s, dA)

$V = \boldsymbol{s} \,.\, \boldsymbol{dA}$

$dQ = n\,Q\boldsymbol{s} \,.\boldsymbol{dA} = n\,\boldsymbol{p}\,.\boldsymbol{dA} = \boldsymbol{P}\,.\,\boldsymbol{dA} = P_\perp .dA$

$$P_\perp = \frac{dQ}{dA} = \sigma_{bound} \quad [\mathrm{C/m^2}]$$

Internal polarization shows itself as bound surface charge

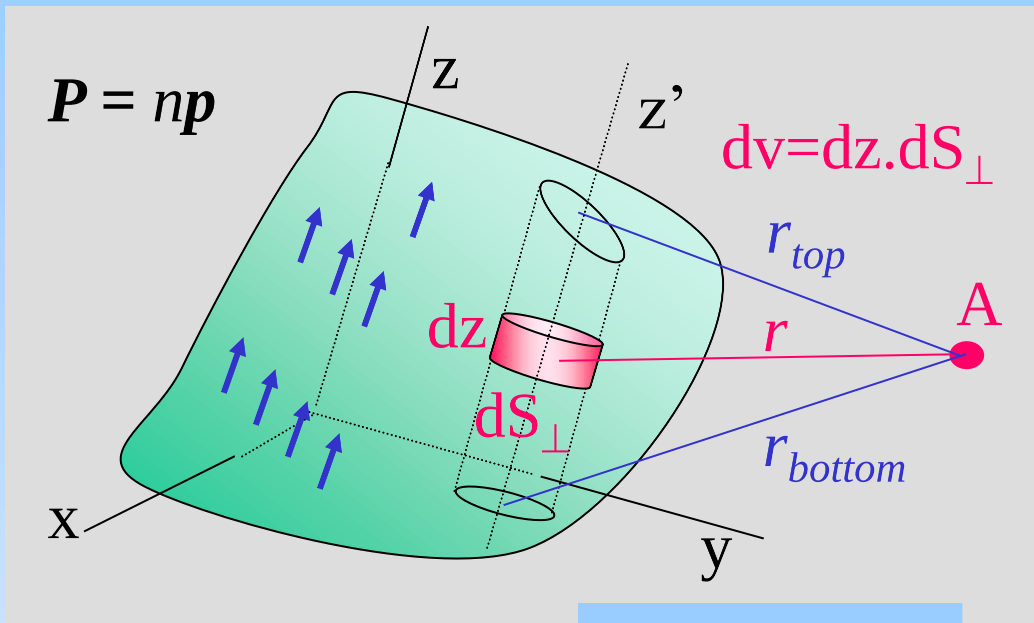
27. Volume polarization ⇒ Surface Charge

<u>Suppose</u>: n dipoles/m^3 ; each dipole moment $\boldsymbol{p}$

<u>Assume</u>: Z-axis // $\boldsymbol{p}$

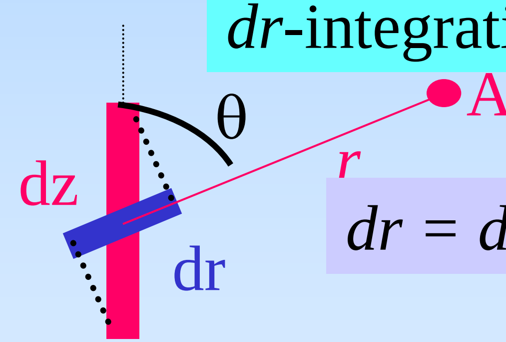
<u>Question</u>: calculate V in A

with $dV = \dfrac{d\,p.\cos\theta}{4\pi\varepsilon_0 r^2}$



<u>Approach</u>: replace *dz*-integration with *dr*-integration

$P.dS_\perp = \boldsymbol{P.dS}$



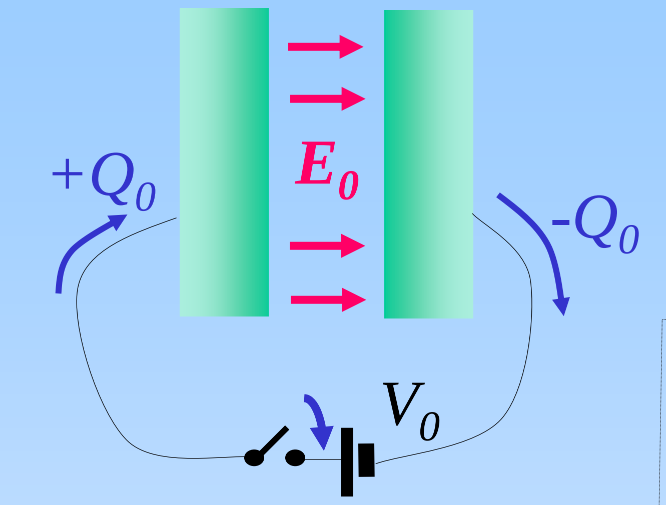
$dr = dz.\cos\theta$

$dp = np.dv = np.dS_\perp.dz = P.dS_\perp.dr / \cos\,\theta$

$$V_A = \iint_S dS_\perp \int_{r_{bottom}}^{r_{top}} \frac{P.dr}{4\pi\varepsilon_0 r^2} = \iint_S \frac{\boldsymbol{P} \bullet \boldsymbol{dS}}{4\pi\varepsilon_0}\left[\frac{1}{r_{bottom}} - \frac{1}{t_{top}}\right]$$

⇒ Polarization => surface charges !!

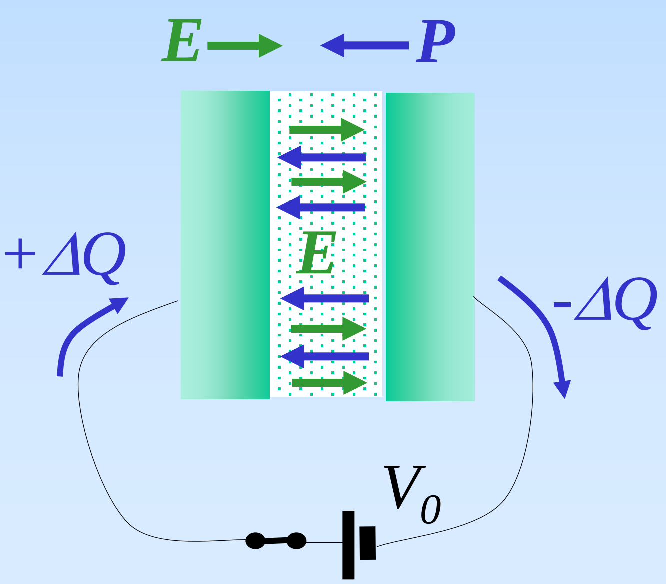
28. Dielectric Displacement



1. Upon closing switch: Plates will be charged: $\pm Q_0$

Between plates: no flow of charges, but ***E***-field present

Consider field as representing imaginairy flow $\boldsymbol{D_0}$ of charges: $\boldsymbol{D_0} = \varepsilon_0 \boldsymbol{E_0} = Q_0 / S$ [C/m^2]



2. Insert dielectric, with V_0 = const.

$\Rightarrow$ E remains const. $\Rightarrow$ extra charge ΔQ needed

Now in dielectric: real flow of (bound polarization) charges $\Delta Q = P.S$

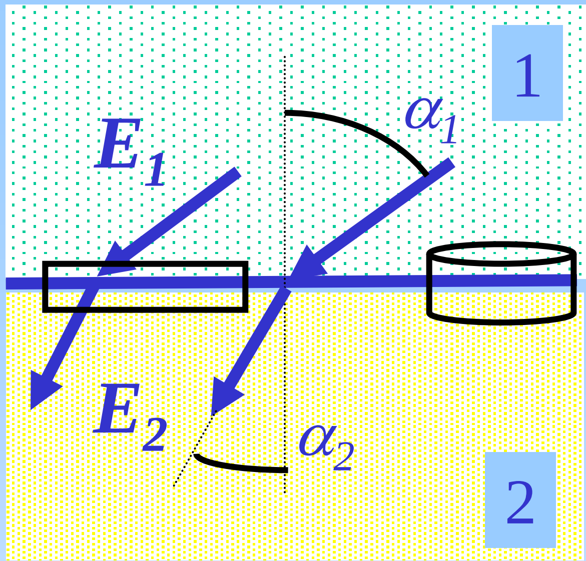
Total charge delivered by battery: $Q = Q_0 + \Delta Q$

$$E_0 = E \quad \Rightarrow \quad \frac{\sigma_0}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0} - \frac{\Delta\sigma}{\varepsilon_0}$$

↑ Field ↑ Free Charges ↑ Bound

$$\boldsymbol{E}\ (= \boldsymbol{E}_0) = \frac{\boldsymbol{D} - \boldsymbol{P}}{\varepsilon_0}$$

29. *E*-field at interface



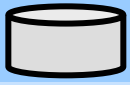
Given: $\boldsymbol{E_1}$; ε_1 ; ε_2

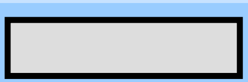
Question: Calculate $\boldsymbol{E_2}$

Needed: "Interface-crossing relations":

$$\oint \boldsymbol{E} \bullet \boldsymbol{dl} = 0 \quad \text{and} \quad \oiint \boldsymbol{D} \bullet \boldsymbol{dA} = Q_{free,\, encl.}$$

Relation *D* and *E*: $\boldsymbol{D} = \varepsilon_0 \, \varepsilon_r \, \boldsymbol{E}$

Gauss box  (empty!): $D_1.A\cos\alpha_1 - D_2.A\cos\alpha_2 = 0 \Rightarrow D_{1\perp} = D_{2\perp}$

Circuit:  $E_1.L\sin\alpha_1 - E_2.L\sin\alpha_2 = 0 \Rightarrow E_{1//} = E_{2//}$

$$\frac{D_1}{E_1}\frac{1}{\tan\alpha_1}\frac{A}{L} = \frac{D_2}{E_2}\frac{1}{\tan\alpha_2}\frac{A}{L}$$

$$\frac{\varepsilon_{r1}}{\varepsilon_{r2}} = \frac{\tan\alpha_1}{\tan\alpha_2}$$

30. Electret: *D* and *E*

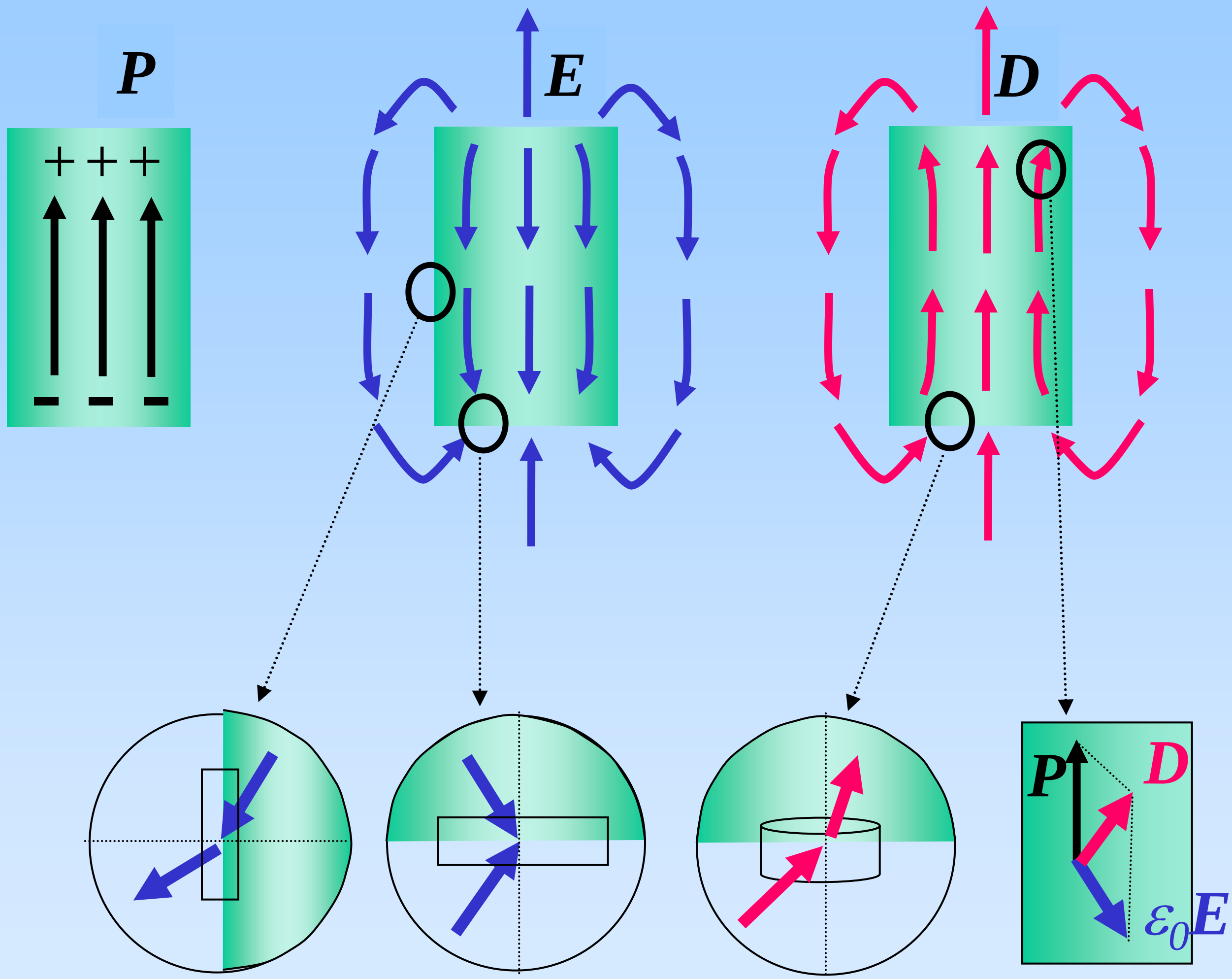
Electret

$\boldsymbol{E} = (\boldsymbol{D}\text{-}\boldsymbol{P}) \,/\, \varepsilon_0$

$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

$$\oint \boldsymbol{E} \bullet d\boldsymbol{l} = 0$$

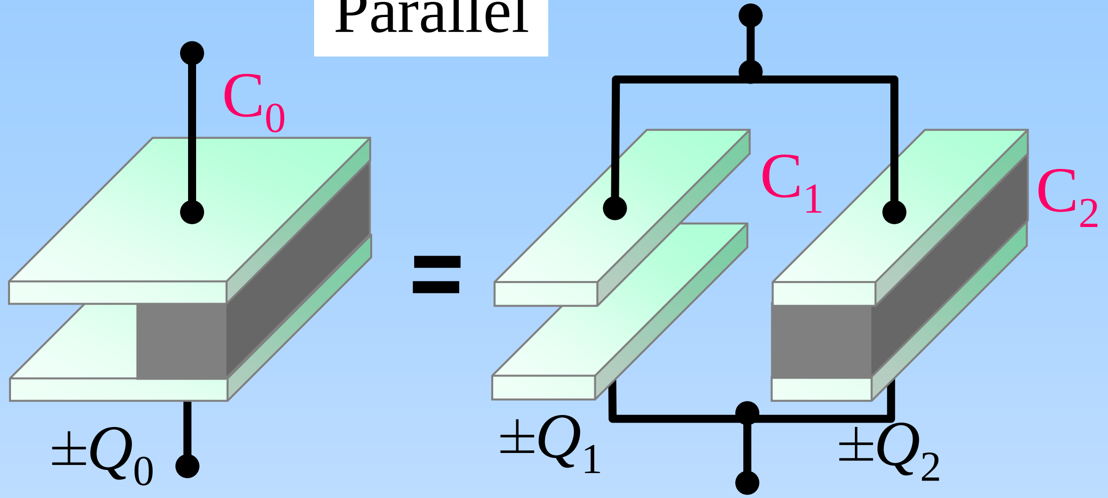
$$\oiint \boldsymbol{D} \bullet d\boldsymbol{S} = Q_f \;\; (=0)$$



D and ***E***·not always parallel

31. Capacitors: series and parallel

Parallel



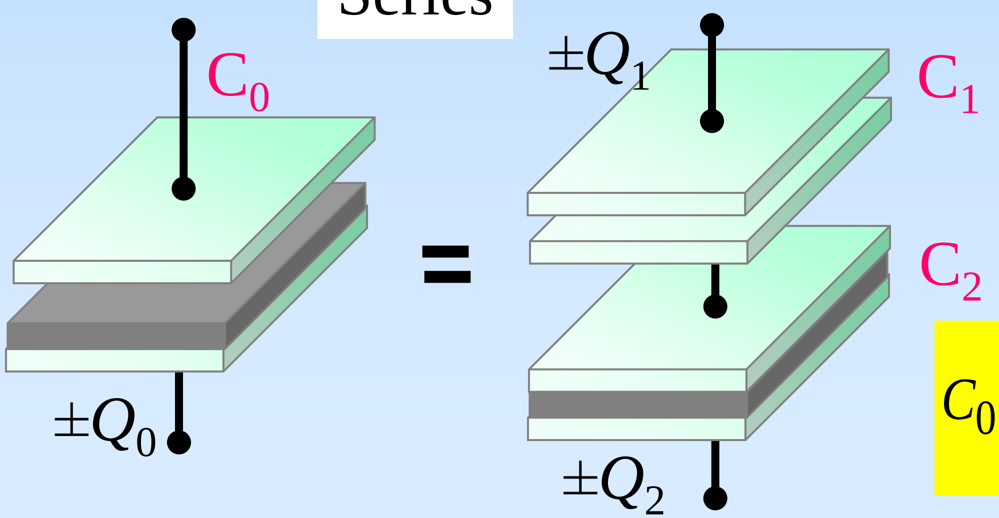
C_1 and C_2 : same V

$$V_1 = V_2 = V_0$$

$$Q_1 + Q_2 = Q_0$$

$$C_0 = C_1 + C_2$$

Series



C_1 and C_2 : same Q

$$Q_1 = Q_2 = Q_0$$

$$V_1 + V_2 = V_0$$

$$C_0 = \frac{Q_0}{V_0} = \frac{Q_0}{V_1 + V_2} \quad \Rightarrow \quad \frac{1}{C_0} = \frac{1}{C_1} + \frac{1}{C_2}$$

32. Capacitors: spheres and cylinders