University of Twente
Department Applied Physics

First-year course on

Electromagnetism

Electromagnetic Waves : Topics

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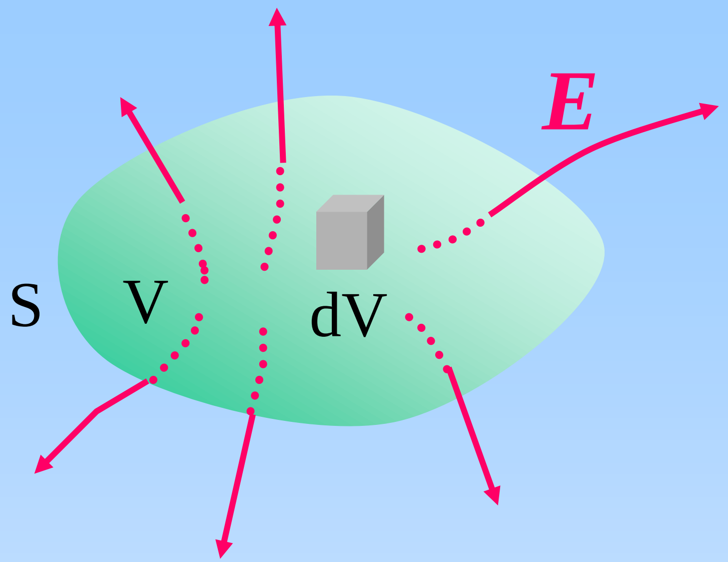
Presentations:

- Electromagnetism: History
- Electromagnetism: Electr. topics
- Electromagnetism: Magn. topics
- Electromagnetism: Waves topics
- Capacitor filling (complete)
- Capacitor filling (partial)
- Divergence Theorem
- E-field of a thin long charged wire
- E-field of a charged disk
- E-field of a dipole
- E-field of a line of dipoles
- E-field of a charged sphere
- E-field of a polarized object
- E-field: field energy
- Electromagnetism: integrations
- Electromagnetism: integration elements
- Gauss' Law for a cylindrical charge
- Gauss' Law for a charged plane
- Laplace's and Poisson's Law
- B-field of a thin long wire carrying a current
- B-field of a conducting charged sphere
- B-field of a homogeneously charged sphere

Electromagnetic Waves

Contents:

1. Gauss' and Faraday's Laws for *E*

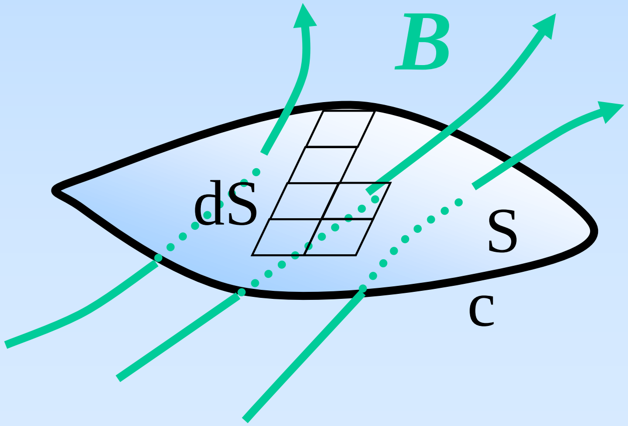


Gauss' Law :

$$\oiint_S \mathbf{E}\bullet\mathbf{dS} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV = \frac{1}{\varepsilon_0}\iiint_V div\mathbf{E}.dV$$

$$div\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Div = micro-flux per unit of volume



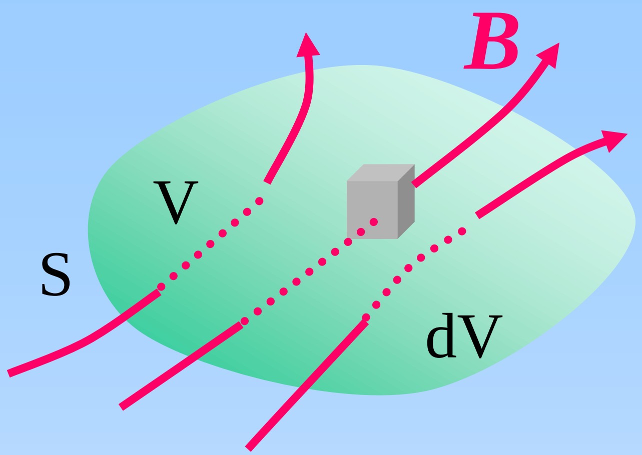
Faraday's Law :

$$V_{ind} = -\frac{d}{dt}\iint_S \mathbf{B}\bullet\mathbf{dS} = \oint_c \mathbf{E_n}\bullet\mathbf{dl} = \iint_S rot\mathbf{E}\bullet\mathbf{dS}$$

$$rot\mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

Rot = micro-circulation per unit of area

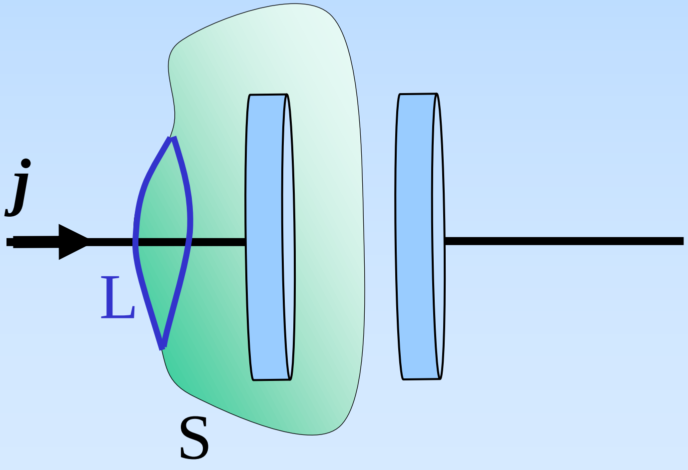
2. Gauss' and Maxwell's Laws for *B*



Gauss' Law :

$$\oiint_S \mathbf{B}\bullet \mathbf{dS} = \iiint_V div\mathbf{B}.dV = 0$$

$$div\ \mathbf{B} = 0$$



Maxwell's Fix for Ampere's Law :

$$\oint_L \mathbf{B}\bullet \mathbf{dl} = \mu_0 \iint_S \mathbf{j}\bullet \mathbf{dS} + \varepsilon_0 \mu_0 \frac{d}{dt} \iint_S \mathbf{E}\bullet \mathbf{dS}$$

$$rot\ \mathbf{B} = \mu_{\mathbf{0}}\mathbf{j} + \varepsilon_0 \mu_0 \frac{d\mathbf{E}}{dt}$$

$$rot\ \mathbf{H} = \mathbf{j}_f + \frac{d\mathbf{D}}{dt}$$

3. Maxwell's Equations and Wave Equation

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \qquad \nabla \bullet \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{j}_f + \frac{d\mathbf{D}}{dt}$$

In vacuum ($\rho = 0$ and $\boldsymbol{j} = 0$): $\nabla \bullet \boldsymbol{E} = 0$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{d\mathbf{B}}{dt} = -\frac{d}{dt}(\nabla \times \mathbf{B}) = \ldots$$

$$\ldots = -\mu_0 \frac{d}{dt}(\nabla \times \mathbf{H}) = -\mu_0 \frac{d^2\mathbf{D}}{dt^2} = -\varepsilon_0 \mu_0 \frac{d^2\mathbf{E}}{dt^2}$$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} + \nabla(\nabla \bullet \mathbf{E}) = -\nabla^2 \mathbf{E}$$

$$\Bigg\} \quad \nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \frac{d^2\mathbf{E}}{dt^2}$$

This is a 3-Dimensional Wave Equation

General form:

v = velocity:

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \ldots. = \frac{1}{v^2} \frac{\partial^2 A}{\partial t^2}$$

$$v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

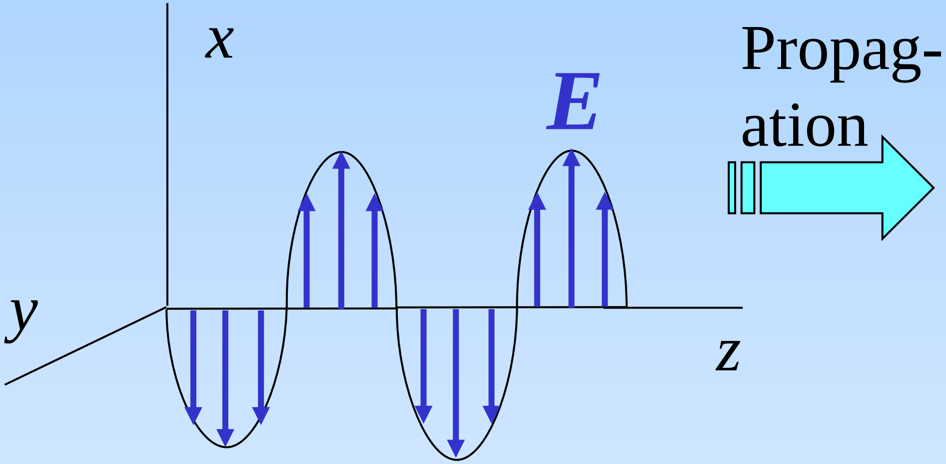
$v = 2.99\ldots \times 10^8$ m/s = light velocity

4. Harmonic Solution of theWave Equation

$$\nabla^2\mathbf{E} = \varepsilon_0\mu_0 \frac{d^2\mathbf{E}}{dt^2}$$

E may have 3 components: $\boldsymbol{E_x}$ $\boldsymbol{E_y}$ $\boldsymbol{E_z}$
Choose X-axis // ***E*** $\Rightarrow$ $E_y = E_z = 0$
("**Polarization direction**" = X-axis).

Question: Does a plane-wave expression for $\boldsymbol{E_x}$ satisfy the wave equation?



$\boldsymbol{E_x} = \boldsymbol{E_{x0}} \exp\{i\,(\omega t\text{-}kz)\}$:

$\boldsymbol{E_{x0}}$ = amplitude + polarization vector;
+Z-axis = direction of propagation

NB. $\boldsymbol{E_x}$ has a similar form for all *y*-values !! (like rolling sea waves)

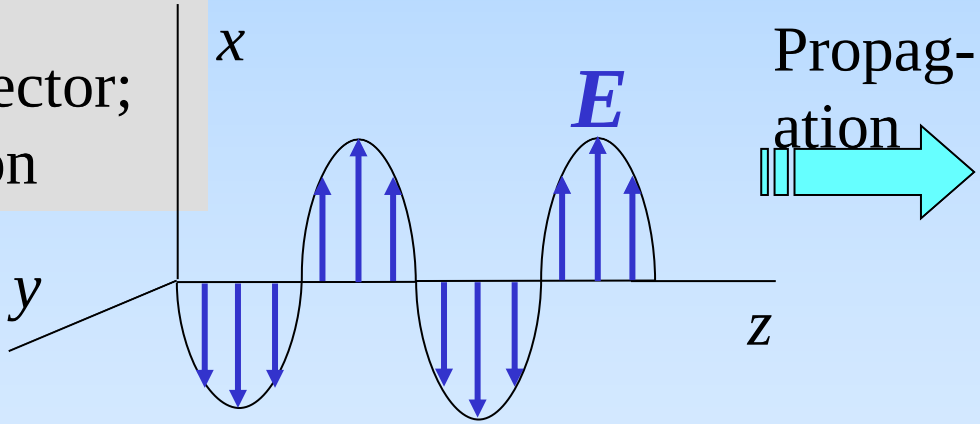
4. Harmonic Solution of the Wave Equation: Plane Waves

$$\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \frac{d^2 \mathbf{E}}{dt^2}$$

E may have 3 components: $\boldsymbol{E_x}$ $\boldsymbol{E_y}$ $\boldsymbol{E_z}$
Choose X-axis // ***E*** $\Rightarrow$ $E_y = E_z = 0$
(Polarization direction = X-axis).

Does a plane-wave expression for $\boldsymbol{E_x}$ satisfy the wave equation?

$\boldsymbol{E_x} = \boldsymbol{E_{x0}} \exp\{i\,(\omega t - kz)\}$:
$\boldsymbol{E_{x0}}$ = amplitude + polarization vector;
+Z-axis = direction of propagation



Insertion into wave equation:

$$\frac{\partial^2 E_x}{\partial z^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2}$$

$$k^2 E_x = \varepsilon_0\, \mu_0\, \omega^2\, E_x \;\; = (\omega^2/c^2) E_x$$

$k = \omega / c = 2\pi / \lambda$
k = wave number ; λ = wavelength
(in 3D-case: ***k*** = wave vector)

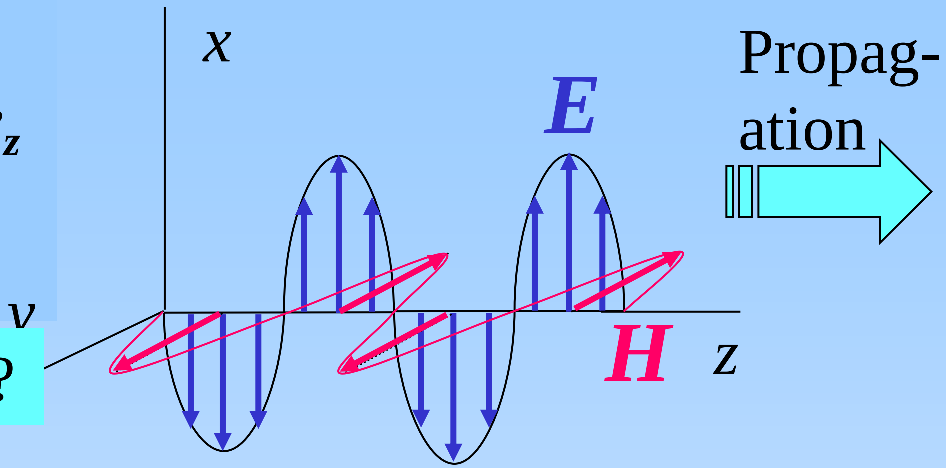
5. Plane waves (1): orientation of fields

Suppose: $\boldsymbol{E}$ // x-axis;

.. propagation // +z-axis: $\boldsymbol{k} // \boldsymbol{e_z}$

.. $\boldsymbol{E_x} = \boldsymbol{E_{x0}} \exp i\,(\omega t - kz)$

.. $\boldsymbol{E_y} = \boldsymbol{E_z} = 0$



Question: what is direction of $\boldsymbol{B}$?

Remember: $$div\ \mathbf{A} = \nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

In *div* and *rot*: only $\partial/\partial z \neq 0$, due to: $\exp i\,(\omega t\text{-}kz)$!

(1) $div\ \boldsymbol{E} = 0$ $\Rightarrow$ $-ik\ \boldsymbol{e_z} . \boldsymbol{E} = 0$ $\Rightarrow$ $\boldsymbol{E}, \boldsymbol{D} \perp \boldsymbol{e_z}$

(2) $div\ \boldsymbol{B} = 0$ $\Rightarrow$ $-ik\ \boldsymbol{e_z} . \boldsymbol{H} = 0$ $\Rightarrow$ $\boldsymbol{H}, \boldsymbol{B} \perp \boldsymbol{e_z}$

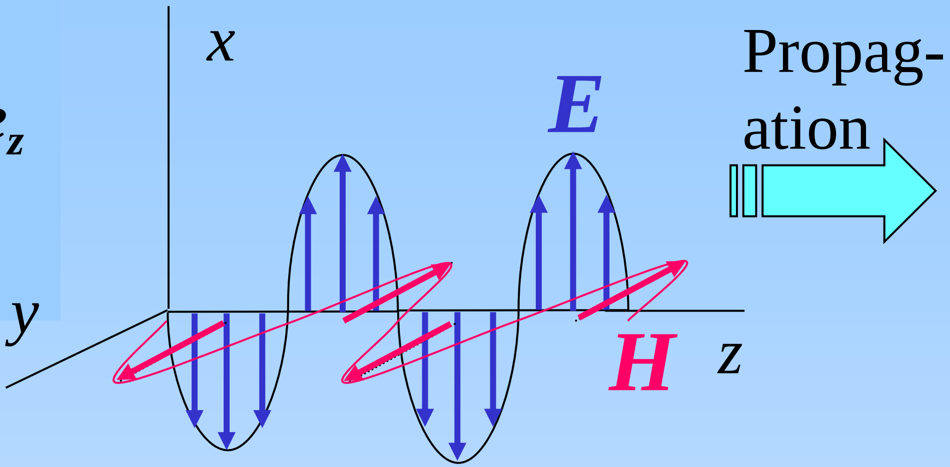
5. Plane waves (1): orientation of fields

Suppose: $\boldsymbol{E}$ // x-axis;

.. propagation // +z-axis: $\boldsymbol{k}$ // $\boldsymbol{e}_z$

.. $E_x = E_{x0} \exp i\,(\omega t - kz)$

.. $E_y = E_z = 0$



div $\boldsymbol{E} = 0$ ➔ $\boldsymbol{E},\boldsymbol{D} \perp \boldsymbol{e}_z$

div $\boldsymbol{H} = 0$ ➔ $\boldsymbol{H},\boldsymbol{B} \perp \boldsymbol{e}_z$

What is mutual direction of $\boldsymbol{E},\boldsymbol{D}$ and $\boldsymbol{H},\boldsymbol{B}$?

Remember:

$$rot\,\mathbf{X} = \nabla \times \mathbf{X} = \begin{vmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ \frac{\partial}{\partial \mathbf{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X_x & X_y & X_z \end{vmatrix}$$

In *div* and *rot*: only $\partial/\partial z \neq 0$,
due to: $\exp i\,(\omega t - kz)$!

(3) $rot\,\boldsymbol{E} = -\,\mathrm{d}\boldsymbol{B}/\mathrm{d}t$ $\Rightarrow -\mathrm{i}k\,\boldsymbol{e}_z x\boldsymbol{E} = -\mathrm{i}\omega\mu\boldsymbol{H}$ $\Rightarrow \boldsymbol{E},\boldsymbol{D} \perp \boldsymbol{H},\boldsymbol{B}$

(4) $rot\,\boldsymbol{H} = \boldsymbol{j}_f + \mathrm{d}\boldsymbol{D}/\mathrm{d}t$ $\Rightarrow -\mathrm{i}k\,\boldsymbol{e}_z\,x\boldsymbol{H} = \sigma\boldsymbol{E} + \mathrm{i}\omega\varepsilon\boldsymbol{E}$ $\Rightarrow \boldsymbol{E},\boldsymbol{D} \perp \boldsymbol{H},\boldsymbol{B}$

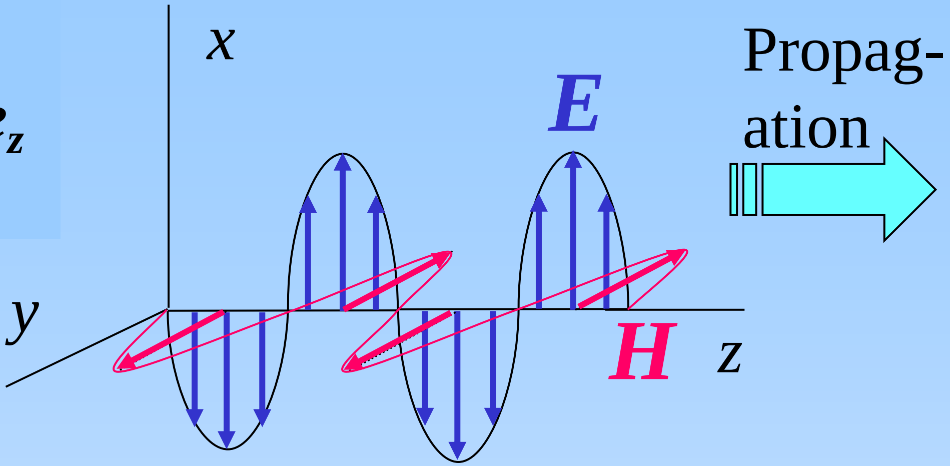
$\boldsymbol{j}_f = \sigma\boldsymbol{E}$

5. Plane waves (1): orientation of fields

Suppose: $\boldsymbol{E}$ // x-axis;

.. propagation // +z-axis: $\boldsymbol{k}$ // $\boldsymbol{e}_z$

.. $\boldsymbol{E}_x = \boldsymbol{E}_{x0} \exp i\,(\omega t - kz)$



(1) $div\,\boldsymbol{E} = 0$

(2) $div\,\boldsymbol{B} = 0$

(3) $rot\,\boldsymbol{E} = -\,\mathrm{d}\boldsymbol{B}/\mathrm{dt}$

(4) $rot\,\boldsymbol{H} = \boldsymbol{j}_f + \mathrm{d}\boldsymbol{D}/\mathrm{dt}$

$\boldsymbol{j}_f = \sigma\boldsymbol{E}$

Consequences:

(1)+(2): $\boldsymbol{E}$ and $\boldsymbol{H} \perp \boldsymbol{e}_z$

(3)+(4): $\boldsymbol{E} \perp \boldsymbol{H}$

If $\boldsymbol{E}$ chosen // x-axis, then $\boldsymbol{H},\boldsymbol{B}$ // y-axis

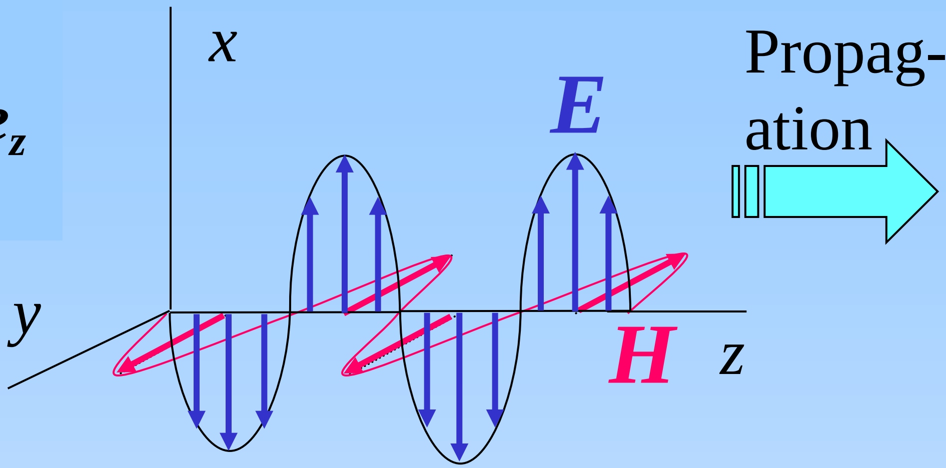
$\boldsymbol{H},\boldsymbol{B}$ also harmonic: $\boldsymbol{B}_y = \boldsymbol{B}_{y0} \exp i\,(\omega t - kz)$

5. Plane waves (2): complex wave vector

Suppose: $\boldsymbol{E}$ // x-axis;

.. propagation // +z-axis: $\boldsymbol{k}$ // $\boldsymbol{e}_z$

.. $E_x = E_{x0} \exp i\,(\omega t - kz)$



(3)⇒ $-ik\,\boldsymbol{e}_z x\boldsymbol{E} = -i\omega\mu\boldsymbol{H}$

(4)⇒ $-ik\,\boldsymbol{e}_z x\boldsymbol{H} = \sigma\boldsymbol{E} + i\omega\varepsilon\boldsymbol{E}$

$$\mathbf{E} = \frac{-ik\,\mathbf{e}_z \times \mathbf{H}}{\sigma + i\omega\varepsilon} = \frac{-ik\,\mathbf{e}_z \times (k\mathbf{e}_z \times \mathbf{E})}{(\sigma + i\omega\varepsilon).\omega\mu}$$

and with:

$\mathbf{e}_z \times \mathbf{e}_z \times \mathbf{E} = -\mathbf{E}$

$$\mathbf{E} = \frac{+ik^2\,\mathbf{E}}{(\sigma + i\omega\varepsilon).\omega\mu}$$

$ik^2 = (\sigma + i\omega\varepsilon).\omega\mu$

→ k complex: $k = k_{Re} + ik_{Im}$

$$\exp(-ikz) = \underbrace{\exp(-ik_{Re}\,z)}\ .\ \underbrace{\exp(k_{Im}\,z)}$$

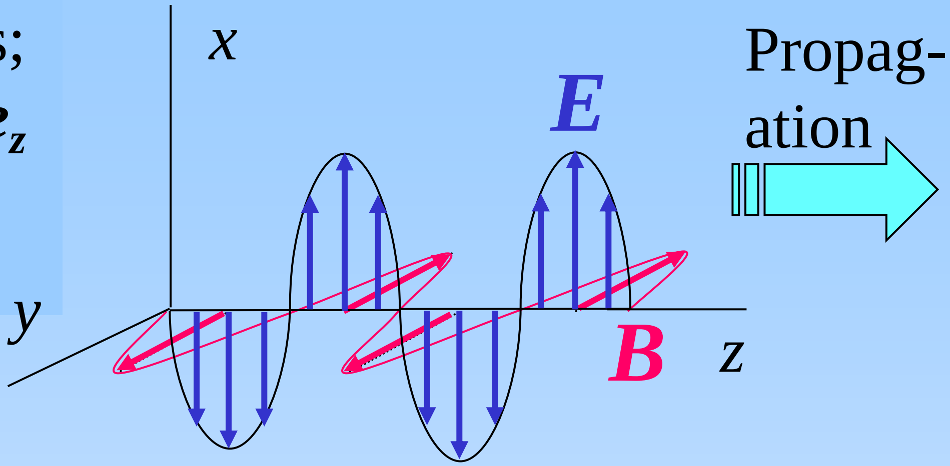
harmonic

$k_{Im} < 0$: absorption

> 0 : amplification (“laser”)

5. Plane waves (3): ***B-E*** correspondence

Suppose: ***E*** // x-axis; ***B*** // y-axis;
.. propagation // +z-axis: $\boldsymbol{k} \, // \, \boldsymbol{e}_z$
.. $\boldsymbol{E}_x = \boldsymbol{E}_{x0} \exp i\,(\omega t - kz)$
.. $\boldsymbol{B}_y = \boldsymbol{B}_{y0} \exp i\,(\omega t - kz)$



Faraday: *rot* ***E*** = - d***B***/dt

$$\begin{pmatrix} \mathbf{e}_\mathbf{x} & \mathbf{e}_\mathbf{y} & \mathbf{e}_\mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{pmatrix} = -\frac{\partial}{\partial t} \begin{pmatrix} 0.\mathbf{e}_\mathbf{x} \\ B_y.\mathbf{e}_\mathbf{y} \\ 0.\mathbf{e}_\mathbf{z} \end{pmatrix}$$

Y - components are ≠ 0 only :

$$-\tfrac{\partial}{\partial z} E_x = -\tfrac{\partial}{\partial t} B_y \Rightarrow -kE_{0x} = -\omega B_{0y}$$

$$\text{Result :} \quad E_0 = \frac{\omega}{k} B_0 = cB_0$$

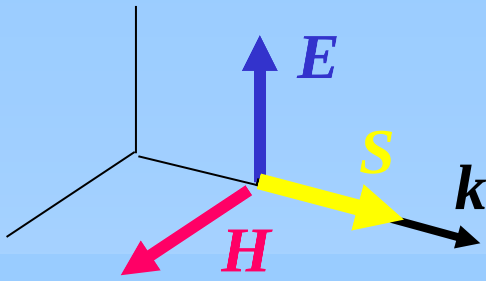
Maxwell (for ***j***=0): *rot* ***H*** = d***D***/dt:
similar result

Maxwell equations are (partly) redundant.

6. The Poynting vector $\boldsymbol{S}$ and Intensity

Definition (for free space) : $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$

Direction of $\boldsymbol{S}$: // $\boldsymbol{k}$



$$\nabla\bullet \boldsymbol{S} = \nabla\bullet(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{H}\bullet(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\bullet(\nabla\times\boldsymbol{H}) =$$

$$= \boldsymbol{H}\bullet(-\mathrm{d}\boldsymbol{B}/\mathrm{dt}) - \boldsymbol{E}\bullet(\boldsymbol{j}_f+\mathrm{d}\boldsymbol{D}/\mathrm{dt}) = -\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t} - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t} - \mathbf{E}\bullet\mathbf{j}_f$$

Integrate over wave volume V (with surface A) and apply Divergence Theorem

$$\underbrace{\oint\!\!\!\oint_A \mathbf{S}\bullet d\mathbf{A}}_{\text{Outflux of energy [J/s] = [W]}} = \iiint_v (\nabla\bullet\mathbf{S})\,dv = -\tfrac{1}{2}\underbrace{\frac{d}{dt}\iiint_v\left[\frac{B^2}{\mu_0}+\varepsilon_0 E^2\right]dv}_{\text{Loss of Electromagnetic field energy [J/s]}} - \underbrace{\iiint_v(\mathbf{E}\bullet\mathbf{j}_\mathbf{f})\,dv}_{\text{Joule heating losses [J/s]}}$$

$\boldsymbol{S}$ = energy outflux per m^2 = Intensity [W/m^2]

The end

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